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An index of (absolute) correlation aversion: theory and some implications

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Abstract

The concept of absolute risk aversion proposed by K. Arrow (1965) and J. Pratt (1964) and the assumption that it is decreasing in wealth has played a central role in the analysis of risky choices. Ten years later S. Richard (1975) defined correlation aversion in the framework of bivariate utility functions. Surprisingly however the measure of the intensity of correlation aversion has received so far almost no attention. In this paper we define an index of (absolute) correlation aversion and stress some of its properties. Besides we show how the assumption that it is decreasing in wealth generates new results for the analysis of risky choices under bivariate utility. Finally we indicate how these notions can be extended to higher orders of risk attitudes.

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Introduction

Much progress in the analysis of risk resulted from the presentation of the index of absolute risk aversion by Arrow (1965) and Pratt (1964) and from the assumption that it is decreasing in wealth. A few years later Rothschild and Stiglitz (1970, 1971) complemented the Arrow-Pratt's contribution by introducing the notion of a mean preserving increase in risk. The combination of a property of the utility (Arrow-Pratt) and of a statistical definition (Rothschild-Stiglitz) was at the origin of a huge theoretical and empirical literature about risk in many fields (economics, finance, operations research).

Quite interestingly, the notion of correlation aversion – that is more general – appeared a little later than that of risk aversion. The first paper on the topics was written by S. Richard (1975) who showed the link between correlation aversion and the sign of the second cross derivative of a bivariate utility function[1]. A few years later, Epstein and Tanny (1980) proposed the extension of the notion of a mean preserving change in risk to a bivariate environment.

Although both literatures present many similarities they also exhibit a major difference. Contrarily to what happened for the index of absolute risk aversion, that was specified from the very beginning, in Arrow and Pratt, no such intensity measure was developed for the phenomenon of correlation aversion.

The purpose of the present paper is precisely first to present an index of absolute correlation aversion (*i.e.* a second order notion) and then to extend it to the third order by proposing an index of cross downside risk aversion (also called cross prudence).
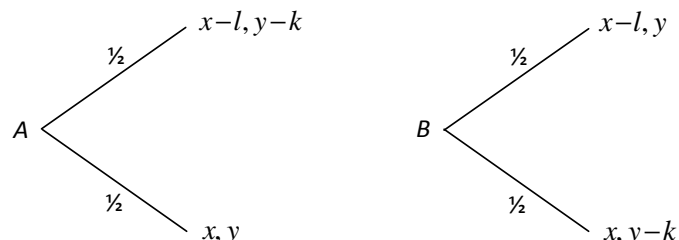
For each of these two indices we then make the assumption that they are decreasing in wealth as Arrow-Pratt and Kimball (1991) did for the absolute risk aversion and absolute prudence coefficients. Such an assumption is quite natural: decision makers dislike positive correlation but this corelation aversion should become less problematic as the agent's wealth increases. Once this assumption is admitted, we can discuss with new arguments the implications for optimal portfolio composition of changes in other attributes of the investor's utility function.

Our paper is organized as follows. In the first section we quickly review the notion of correlation aversion and cross prudence. We then propose an index for each of these concepts and we discuss the assumption that they are decreasing in wealth. In the third section we derive some implications for portfolio composition.

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[1] Notice the relationship with risk aversion which is linked to the sign of the second derivative of a univariate utility function.

1   Correlation aversion and cross prudence

To the best of our knowledge the first paper on correlation aversion was published in 1975 by Scott Richard. Essentially, this author compares two binary lotteries.

In these lotteries x and y are two attributes (*e.g.* x is wealth and y is health). These attributes are subject to a potential loss ($-l$ for the first attribute and $-k$ for the second one). The difference between A and B results from the fact that in A the losses are perfectly positively correlated while in B they are perfectly negatively correlated[2]. For an expected utility decision maker, the difference between A and B amounts to:

$$
\begin{aligned}
R &= E[u(B)] - E[u(A)] \\
&= \frac{1}{2}[(u(x-l,y) + u(x,y-k)) - (u(x-l,y-k) + u(x,y))]
\end{aligned}
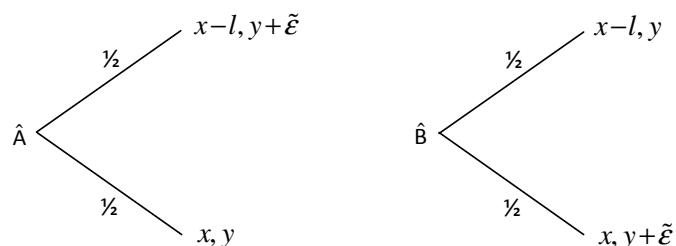\tag{1.1}
$$

and it is easily shown that

$$
R \gtreqless 0 \iff u_{12}(x,y) \lesseqgtr 0
\tag{1.2}
$$

Indeed, expanding each term in (1.1) up to the second order around $u(x,y)$ yields:

$$
R = E[u(B)] - E[u(A)] \simeq \frac{lk}{2}(-u_{12}(x,y))
\tag{1.3}
$$

Hence the idea of correlation aversion is expresses under expected utility through a negative sign of the second cross derivative of u.

To define cross prudence[3], the loss $-k$ in terms of the 2$^{\text{nd}}$ attribute is replaced by the attachment of a zero mean risk applied to that attribute. Hence one compares loteries $\widehat{A}$ and $\widehat{B}$.

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[2] For an extension to intermediate changes in the degree of correlation see Denuit and Rey (2010).

[3] More details on this and extensions to higher orders can be found in Eeckhoudt, Rey and Schlesinger (2007).

As for loteries A and B, pains are perfectly positively correlated in $\widehat{A}$ and perfectly negatively correlated in $\widehat{B}$. It is then shown that for expected utility decision makers:

$$E[u(\widehat{B})] - E[u(\widehat{A})] \gtreqless 0 \Longleftrightarrow u_{122} \gtreqless 0 \tag{1.4}$$

This result is obtained by considering $\widehat{R}$:

$$
\begin{aligned}
\widehat{R} &= E[u(\widehat{B})] - E[u(\widehat{A})] \\
&= \frac{1}{2}[(u(x-l,y) + E[u(x,y+\widetilde{\epsilon})]) - (E[u(x-l,y+\widetilde{\epsilon})] + u(x,y))]
\end{aligned}
\tag{1.5}
$$

Expanding up to the 2nd order each term involving $\widetilde{\epsilon}$ we obtain:

$$\widehat{R} \simeq \frac{1}{2}[(u(x-l,y) + u(x,y) + \frac{\sigma_{\widetilde{\epsilon}}^2}{2}u_{22}(x,y) - (u(x-l,y) + \frac{\sigma_{\widetilde{\epsilon}}^2}{2}u_{22}(x-l,y) + u(x,y))]$$

so that

$$
\begin{aligned}
\widehat{R} &\simeq \frac{1}{2}\frac{\sigma_{\widetilde{\epsilon}}^2}{2}[u_{22}(x,y) - u_{22}(x-l,y)] \\
&\simeq \frac{\sigma_{\widetilde{\epsilon}}^2}{4}lu_{122}(x,y)
\end{aligned}
\tag{1.6}
$$

When u_{122} is positive, *i.e.* when there is correlation aversion at the 3rd order, the agent is said to be cross prudent.

While attention has been paid so far in the literature to the signs of the cross derivatives and hence to the directions of preferences, we now propose in section 2 an intensity measure that extends to the bivariate case the indices of absolute risk aversion and of absolute downside risk aversion.

2 Indices of correlation aversion

Let us start with the index at the second order. Since $u_{12} < 0$ induces that B is prefered to A (see (1.2)), let us raise the following question: what is the amount of money (m) to be paid to the decision maker so that he becomes indifferent between A and B ? In a sense m measures the willingness to accept a perfectly positive correlation when initially the agent faces a perfectly negative one. Hence we define m by:

$$\frac{1}{2}[(u(x-l,y) + E(u(x,y-k)) = \frac{1}{2}[(u(x-l,y-k)) + u(x+m,y)] \tag{2.1}$$

Approximating $u(x+m,y)$ up to the first order around $u(x,y)$ and using the results about $E[u(B)] - E[u(A)]$ of the previous section, we are finally left with:

$$m \simeq \frac{lk(-u_{12}(x,y))}{2u_1(x,y)} \tag{2.2}$$

In this expression $\frac{-u_{12}}{u_1}$ is the index of absolute correlation aversion and it is positive if $u_{12} < 0$.

This result is rather intuitive: in (2.2) we divide R of (1.3) – which is a loss in utility – by the marginal utility of money in order to convert into a monetary equivalent what is initially a utility loss.

At the third order, we proceed in the same way[4]. We add an amount of money $\widehat{m}$ to x in $u(x, y)$ so that $E[u(\widehat{A})]$ becomes equal to $E[u(\widehat{B})]$ and we then have:

$$u(x - l, y) + E\left[u(x, y + \widetilde{\epsilon})\right] = E\left[u(x - l, y + \widetilde{\epsilon})\right] + u(x + \widehat{m}, y) \tag{2.3}$$

Expanding $u(x + \widehat{m}, y)$ around $u(x, y)$ up to the first order and using the results of the previous section, we obtain:

$$\widehat{m} \simeq \frac{\sigma_{\widetilde{\epsilon}}^2 l u_{122}(x, y)}{4 u_1(x, y)} \tag{2.4}$$

In (2.4) $\frac{u_{122}(x,y)}{u_1(x,y)}$ is the index of cross downside risk aversion (or in other terms correlation aversion at the third order).

At this stage, two points are to be stressed:

- if x and y are perfect substitutes, $u(w, y)$ becomes $u(x + y)$ and our bivariate indices reduce to the well known index of absolute risk aversion ($-\frac{u''}{u'}$) and index of downside risk aversion ($\frac{u'''}{u'}$).

- from now on we make the assumption that the two indices of correlation aversion are decreasing in wealth. Such an assumption extends to bivariate utility the well known ones in the univariate case and we examine some of their implications for portfolio choice.

3   Some implications for portfolio choices

We consider the standard problem of an investor who can allocate his wealth between a safe asset (m) and a risky one. The safe asset pays a return i and the risky one has a binary return: x_0 with a probability p and x_1 with a probability $1 - p$. In order to have an interior solution we make the following assumptions:

$$x_0 - i \;\; < \;\; 0 < x_1 - i$$
$$\text{and } E(\widetilde{x}) \;\; > \;\; i$$

If we denote by a the amount invested in the risky asset, the optimization program is written:

$$\underset{a}{\text{Max }} Z = pu(w_0 + a(x_0 - i), y) + (1 - p)u(w_0 + a(x_1 - i), y) \tag{3.1}$$

---

[4]The equivalent approach in the univariate case can be found in Modica and Scarsini (2005) and Crainich and Eeckhoudt (2008)while the extension to higher orders is in Denuit and Eeckhoudt (2010).

where y is as before the second argument of the utility function.

The first-order condition assiciated with (3.1) is:

$$\frac{dZ}{da} = pu_1(x_0 - i, y)(x_0 - i) + (1-p)u_1(x_1 - i, y)(x_1 - i) = 0 \tag{3.2}$$

where $x_0 - i$ or $x_1 - i$ as arguments of u_1 stand respectively for $w_0 + a^*(x_0 - i)$ and $w_0 + a^*(x_1 - i)$ while a^* is the optimal value of a.

The second-order condition is satisfied for $u_{11} < 0$.

In this section we analyze two different questions:

- how does an exogenous change in y affect a^*?

- how does a^* react to the addition of an independent zero mean risk to y ?

3.1 A change in y

If there is an exogenous increase in y we have:

$$\text{sign } \left(\frac{da^*}{dy}\right) = \text{sign } \frac{\partial}{\partial y}\left(\frac{dZ}{da}\right)$$

so that:

$$\text{sign } \left(\frac{da^*}{dy}\right) = pu_{12}(x_0, y)(x_0 - i) + (1-p)u_{12}(x_1, y)(x_1 - i) \tag{3.3}$$

and since $(x_0 - i)$ is negative while $(x_1 - i)$ is positive, it is easy to show that:

$$\left(\frac{da^*}{dy}\right) \gtreqqless 0 \text{ if } \frac{u_{12}(x_0, y)}{u_1(x_0, y)} \lesseqqgtr \frac{u_{12}(x_1, y)}{u_1(x_1, y)}$$

or

$$\left(\frac{da^*}{dy}\right) \gtreqqless 0 \text{ if } \frac{\partial}{\partial x}\left(\frac{u_{12}(x, y)}{u_1(x, y)}\right) \gtreqqless 0 \tag{3.4}$$

Keeping in mind that the index of correlation aversion equals $-\frac{u_{12}}{u_1}$ we can conclude that under decreasing correlation aversion an improvement in the second argument of u raises the demand for the risky asset. For instance if y represents health we obtain that ceteris paribus improved health stimulates financial risk taking.

At this stage, it should be stressed that properties of the optimal choices in a bivariate model have been often analyzed in the past. However the analysis was done without reference to the index of correlation aversion which offers new insights. For instance since health deteriorates with age, an index of correlation aversion that decreases with wealth can explain the fall on risk taking for older people.

3.2 A background risk on y

If y becomes $y + \widetilde{\epsilon}$ where $\widetilde{\epsilon}$ is an independent zero mean background risk, it can be shown that the new optimal value of a (a^{**}) will be lower than the initial one (a^*) if $\frac{u_{122}}{u_1}$, the index of correlation aversion at the third order, is decreasing in wealth.

To prove this result, let's first consider the objective function $\widehat{Z}$ with the background risk.

$$\widehat{Z} = pE[u(w_0 + a(x_0 - i), y + \widetilde{\epsilon})] + (1-p)E[u(w_0 + a(x_1 - i), y + \widetilde{\epsilon})] \qquad (3.5)$$

We now evaluate $\frac{d\widehat{Z}}{da}$ at a^* and we find:

$$\frac{dZ}{da}\big|_{a^*} = pE[u_1(x_0 - i, y + \widetilde{\epsilon})](x_0 - i) + (1-p)E[u_1(x_1 - i, y + \widetilde{\epsilon})](x_1 - i) = 0 \qquad (3.6)$$

For small risks, this expression becomes:

$$\frac{dZ}{da}\big|_{a^*} = p(x_0 - i)[u_1(x_0 - i, y) + \frac{\sigma^2}{2}u_{122}(x_0 - i, y)] + (1-p)(x_1 - i)[u_1(x_1 - i, y) + \frac{\sigma^2}{2}u_{122}(x_1 - i, y)] = 0$$

which reduces to:

$$\frac{dZ}{da}\big|_{a^*} = \frac{\sigma^2}{2}[p(x_0 - i)u_{122}(x_0 - i, y) + (1-p)(x_1 - i)u_{122}(x_1 - i, y)] = 0 \qquad (3.7)$$

because of the first-order condition for a^*.

Using standards arguments:

$$a^{**} \gtreqless a^* \text{ if the expression in (3.7) is } \gtreqless 0$$

Now, again because the opposite signs of $(x_0 - i)$ and $(x_1 - i)$ we have:

$$a^{**} \gtreqless a^* \text{ if } \frac{u_{122}(x_0 - i, y)}{u_1(x_0 - i, y)} \lesseqgtr \frac{u_{122}(x_1 - i, y)}{u_1(x_1 - i, y)}$$

Hence if $\frac{u_{122}}{u_1}$ is decreasing in wealth the introduction of a zero mean risk on y reduces financial risk taking.

This result nicely parallels those obtained in the univariate case which point out to a substitution between the background risk and the endogenous one.

4 Conclusion

Many papers have been devoted to the definition of the risk premium in a multivariate environment[5]. In this paper we have followed a different approach. While the definition of a risk premium compares a risky situation with a fully sure one, we have maintained here a

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[5] An extensive review of these articles would be the topics for a paper of its own.

comparison between two risky situations (as in (2.1) and (2.3). In this way we have obtained (absolute) indices of correlation aversion and cross downside risk aversion.

We have shown how these indices are related to the basic preference of a decision maker as expressed in section 1. Furthermore some first implications for the analysis of portfolio choices have been developed (section 3).

An interesting extension would be to extend to any risk the developments made in section 3 for small ones. In this way a link could be created with the interesting notion of "cross risk vulnerability" proposed Malvergne and Rey (2009).

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