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A Note on Parameterizing Input Distance Functions: Does the Choice of a Functional Form Matter?

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Abstract: We use a Monte Carlo experiment to compare the quadratic and translog functional forms in terms of their ability to approximate known frontiers that possess convex curvature. Unlike some of the existing simulation studies that have studied this topic, we find that both functional forms provide a reliable approximation to a true frontier. Our results lend support to existing explanations concerning the translog form's innate propensity to yield convex, rather than concave, frontier estimates.

Keywords: Distance Functions, Parameterization

JEL Classification: D24, C63

Two recent simulation studies comparing the ability of distance functions to represent production technology have concluded that quadratic directional output distance function (Chambers et al., 1996, 1998) fares better than translog Shephard (1970) output distance function. Färe et al. (2010) performed this comparison in the quantity space, whereas Chambers et al. (2013) implemented it in the price space. As a possible explanation, both papers mention the translog function's intrinsic propensity to produce frontier estimates that possess convex curvature—a problem when modeling output distance functions, but a useful feature for parameterizing *input* distance functions (Shephard, 1953). In this paper we perform a series of Monte Carlo experiments to compare a quadratic directional *input* distance function to a translog Shephard's *input* distance function in terms of their ability to approximate different families of true production technologies. We rely on the first- and second-order derivatives of these functions in order to determine which of them performs best.

In the next section of this short note we give a quick overview of the two functional forms whose performance we assessed in this study. Section 2 describes our simulation design and our most interesting results. Section 3 concludes.

1. The Functional Form Alternatives

In this section, which builds on Chambers et al. (2013) and Färe et al. (2010), we introduce the functional forms that we consider in this paper.[1] These parametric forms are derived from the two representations of the underlying technology, namely, Shephard's (1953) input distance function and Luenberger's (1992) benefit function, which we term "the directional

[1] We cite these "reference" papers repeatedly throughout the text, due to their importance in motivating the present study.

input distance function” (Chambers et al. 1996), and the condition that they are of the form “generalized quadratic.”

Let $x \in \Re_+^N$ be an input vector and $y \in \Re_+^M$ an output vector, so that the input requirement set in terms of these vectors is

(1) $$L(y) = \{x \in \Re_+^N : x \quad can \quad produce \quad y\}.$$

We assume that $L(y)$ meets the standard properties, such as non-emptiness, closeness, convexity, and disposability. Shephard’s input distance function is defined as $D_i(y,x) = \sup\{\lambda : x/\lambda \in L(y)\}$, and we note that, under weak disposability of inputs,

(*i*) $$D_i(y,x) \geq 1 \Leftrightarrow x \in L(y)$$

and

(*ii*) $$D_i(y,\lambda x) = \lambda D_i(y,x), \quad \lambda > 0,$$

where (i) tells us that the distance function fully represents the technology and (ii) states that it is homogeneous of degree +1 in inputs.

The second representation of $L(y)$ is in terms of the directional input distance function, $\vec{D}_i(y,x;g) = \sup\{\beta : (x - \beta g) \in L(y)\}$, where $g \in \Re_+^N \quad g \neq 0$ is the directional vector, which indicates how *x* is projected towards the boundary of $L(y)$. This distance function satisfies

(*iii*) $$\vec{D}_i(y,x;g) \geq 0 \Leftrightarrow x \in L(y)$$

and

(*iv*) $$\vec{D}_i(y, x-\mu g; g) = \vec{D}_i(y, x; g) - \mu, \quad \mu > 0,$$

i.e. $\vec{D}_i(y, x; g)$, like Shephard's distance function, fully represents the technology. However, it satisfies the translation property (*iv*), rather than homogeneity (*ii*).

Next we show how homogeneity and translation properties influence the choice of functional forms. We say that a function $F : \Re^2 \to \Re$ is a generalized quadratic function if

(2) $$F(q) = \varsigma\left(a_0 + \sum_{i=1}^{2} a_i h(q_i) + \sum_{i=1}^{2}\sum_{j=1}^{2} a_{ij} h(q_i) h(q_j)\right),$$

where $a_i, a_{ij} \in \Re$ and $h, \varsigma : \Re \to \Re$. Färe and Sung (1986) showed that if a function is generalized quadratic and homogeneous it is either

(3) $$F(q) = a_0 + \sum_{i=1}^{2} a_i \ln(q_i) + \sum_{i=1}^{2}\sum_{j=1}^{2} a_{ij} \ln(q_i)\ln(q_j),$$

i.e. translog, or

(4) $$F(q) = \left(a_0 + \sum_{i=1}^{2}\sum_{j=1}^{2} a_{ij} q_i^{r/2} q_j^{r/2}\right)^{1/r},$$

i.e. the quadratic mean of order *r* (Denny 1974, Diewert 1976). Since the latter functional form has only second order terms, we choose the translog function (3) when parameterizing Shephard's input distance function.

If a generalized quadratic function meets the translation property, there exist only two solutions to the resulting functional equation (Färe and Lundberg, 2006), namely[2]

(5) $$F(q) = a_0 + \sum_{i=1}^{2} a_i q_i + \sum_{i=1}^{2}\sum_{j=1}^{2} a_{ij} q_i q_j \ ,$$

i.e. the quadratic function , or

(6) $$F(q) = \frac{1}{2\lambda} \ln \sum_{i=1}^{I}\sum_{j=1}^{I} a_{ij} \exp(\lambda q_i)\exp(\lambda q_j), \quad \lambda \neq 0.$$

Consistent with existing studies that compared the performance of the two distance functions, we will use the quadratic functional form to parameterize the directional input distance function.

2. Monte Carlo Simulation Design and Results

We assume that two inputs produce a single socially desirable output and consider two families of true technologies. Those belonging to the first family have a fifth-order polynomial structure, i.e.

[2] Here we have chosen $g = (1,1)$.

(7) $$L^P(y)=\{(x_1,x_2): x_2^P = \beta_0^P + \beta_1^P x_1 + \beta_2^P x_1^2 + \beta_3^P x_1^3 + \beta_4^P x_1^4 + \beta_5^P x_1^5 + y^{1.5}\}.$$

The second group consists of so-called logarithmic technologies, given by

(8) $$L^L(y)=\{(x_1,x_2): x_2^L = \exp\{\beta_0^L\} x_1^{\beta_1^L} \exp\{y^{1.5}\}\}.$$

We closely follow the design of Färe et al. (2010) and Chambers et al. (2013), since our main goal is to test whether their main conclusions remain unchanged when the true frontier is convex. Varying the vectors of coefficients β^P and β^L enables us to alter the curvature of the true frontier. We consider three such variations for both families of true technologies, labeled by P1, P2, P3, L1, L2, and L3, whose corresponding coefficient values are summarized in Table 1. We choose the coefficients so that the frontiers of P1 and L1 are relatively flat and subsequently change them to add more curvature. Normalizing the outputs with an arbitrary quantity allows us to obtain the plots of these frontiers, similar to those given in Figure 1.

We then randomly draw quantities of the first input and impose assumptions on the parameters of underlying distributions to ensure that the technology is well-behaved. For the polynomial technologies these quantities are generated as $x_1 \sim Gamma(\lambda, \theta)$, and two assumptions are placed on the distribution parameters. Taking $\lambda = 5$ and $\theta = 0.5$ yields relatively unequal quantities of x_1 and x_2 for nearly all observations, whereas assuming $\lambda = 18$ and $\theta = 0.25$ gives relatively balanced input values. In the case of the translog technologies, x_1 is drawn from the uniform distribution on the interval $(0.8, 2.8)$. Both outputs are assumed to be standard uniform and the sample size equals 50, 100, and 500 observations, giving us a total of twenty-seven true models. Finally, we complete the data generating process by assuming that $x_2 = x_2^P + \nu - \varepsilon$ when the true technology is polynomial and $x_2 = x_2^L + \nu - \varepsilon$ when it is translog, where $\nu \sim |\mathrm{N}(0,1)|$ and $\varepsilon \sim \mathrm{N}(0,1)$ represent technical inefficiency and the conventional disturbance term, respectively.

The next step of the experiment involves using these data to estimate distance functions. Since we can assume that $\exp\{\nu\} = D_i(y, x_1, x_2)$, the specification error can be added as $\exp\{\nu\} = D_i(y, x_1, x_2)\exp(\varepsilon)$. After rearranging, plugging this result in the expression representing distance function's homogeneity, assuming $\lambda = x_2^{-1}$, and rearranging again we have the following result:

(9) $$1/x_2 = D_i(y, x_1/x_2, 1)\exp\{\varepsilon - \nu\}.$$

After taking the log of both sides and assuming the translog form for the normalized distance function $\ln(D_i(y, x_1/x_2, 1))$, this specification can be estimated using the Aigner et al. (1977) method.[3] Its estimated parameters can subsequently be used to recover the coefficients of the associated translog Shephard's input distance function $\exp\{\nu\}$:

[3] Relying on Shephard's input distance functions' homogeneity to obtain suitable econometric specifications is common in the literature. See Atkinson et al. (2003a, 2003b) and Grosskopf et al. (1997) for more details.

$$\ln\left(D_i\left(y,x\right)\right)=\gamma_0+\gamma_1\ln\left(y\right)+\left(\gamma_{11}/2\right)\left(\ln\left(y\right)\right)^2+\sum_{i=1}^{2}\delta_i\ln\left(x_i\right)$$

(10)
$$+\left(1/2\right)\sum_{i=1}^{2}\sum_{j=1}^{2}\delta_{ij}\ln\left(x_i\right)\ln\left(x_{i'}\right)+\sum_{i=1}^{2}\omega_i\ln\left(y\right)\ln\left(x_i\right)$$

As far as the directional distance function is concerned, we can first assume that $\nu=\vec{D}_i\left(y,x_1,x_2;g\right)$ and then add the two-sided error to the right-hand side of that equation. Combining this result with the expression for the function's translation property, taking $\mu=x_2/g_2$, and assuming $g=\left(1,1\right)$ yields the following composed-error specification after a series of rearrangements (Färe et al., 2005):[4]

(11)
$$-x_2/g_2=\vec{D}_i\left(y,x_1-x_2/g_2\right)-\nu+\varepsilon\,.$$

This normalized distance function is parameterized using the quadratic functional form and estimated using the same method.[5] Its parameter estimates yield the coefficients of the corresponding quadratic directional distance function ν, given by

(12)
$$\vec{D}_i\left(y,x;g\right)=\varphi_o+\varphi_1 y+\left(\varphi_{11}/2\right)y^2+\sum_{i=1}^{2}\psi_i x_i+\left(1/2\right)\sum_{i=1}^{2}\sum_{j=1}^{2}\psi_{ij}x_i x_j+\sum_{i=1}^{2}\chi_i y x_i$$

[4] Unlike in the previous case, an infinite number of suitable econometric specifications can be obtained by varying the assumptions placed on the mapping vector g, which is located in the third quadrant, suggesting that the inputs are being contracted.

[5] Chambers et al. (2013) relied on the same method, while Färe et al (2013) chose parametric linear programming algorithm of Aigner and Chu (1968).

We use the vectors of maximum-likelihood parameter estimates $\left(\hat{\gamma},\hat{\delta},\hat{\omega}\right)$ and $\left(\hat{\varphi},\hat{\psi},\hat{\chi}\right)$ to recover quadratic and translog input set frontiers in order to assess whether either of these families of estimates fares better than the other. We assume that $\nu=\varepsilon=0$, normalize both outputs with the sample average $\bar{y}$, and use x_1 to get the relevant best-practice quantities of $\hat{x}_2 \equiv x_2\left(\hat{\gamma},\hat{\delta},\hat{\omega}\right)$ and $\tilde{x}_2 \equiv x_2\left(\hat{\varphi},\hat{\psi},\hat{\chi}\right)$ for the quadratic and translog estimates, respectively. Finally, we use the pairs $\left[x_1,\hat{x}_2\right]$ and $\left[x_1,\tilde{x}_2\right]$ together with the distance functions' parameter estimates to compute the associated marginal rate of technical substitution (MRTS) and the Morishima (1967) elasticity of substitution at each observation and then compare them to the true MRTS and elasticity for both parameterizations.[6]

The true MRTS and elasticity can be obtained using the expressions (1) and (2) and are the negative of $\partial x_2/\partial x_1$ and $x_1 \dfrac{\partial^2 x_2/\partial x_1^2}{\partial x_2/\partial x_1}$, respectively, where $x_2 = x_2^P$ for the polynomial technologies and $x_2 = x_2^L$ for the log technologies. The estimated MRTS can be interpreted as the relative shadow price of inputs [Färe and Primont (1995)] and is given by $\dfrac{\partial D(\cdot)/\partial x_1}{\partial D(\cdot)/\partial x_2}$, where

[6] In addition to these two benchmarks, Färe et al. (2010) and Chambers et al. (2013) also compute the Euclidean distance between the true and estimated frontier points. They subsequently average across these three discrepancies before assessing the results using a single criterion, which is based on that average. Here we use only two benchmarks and choose to compare the translog and quadratic functions' ability to approximate the true MRTS separately from elasticity.

$D(\cdot)$ denotes either $\vec{D}_i(y, x_1, x_2; g)$ or $D_i(y, x_1, x_2)$. Consequently, the difference between the estimated and true MRTS is

$$\Omega_k(\hat{\varphi}, \hat{\psi}, \hat{\chi}) = \frac{\partial \vec{D}_i(\bar{y}, x_{1k}, \tilde{x}_{2k}; g) / \partial x_1}{\partial \vec{D}_i(\bar{y}, x_{1k}, \tilde{x}_{2k}; g) / \partial x_2} + \frac{\partial x_{2k}}{\partial x_1} \tag{13}$$

and

$$\Omega_k(\hat{\gamma}, \hat{\delta}, \hat{\omega}) = \frac{\partial \ln \vec{D}_i(\bar{y}, x_{1k}, \hat{x}_{2k}; g) / \partial \ln x_1}{\partial \ln \vec{D}_i(\bar{y}, x_{1k}, \hat{x}_{2k}; g) / \partial \ln x_2} \times \frac{\hat{x}_{2k}}{x_{1k}} + \frac{\partial x_{2k}}{\partial x_1}, \tag{14}$$

where x_{2k} equals either x_{2k}^P or x_{2k}^L, depending on the type of true technology. Our first criterion is based on these discrepancies and is given by $K^{-1}\sum_k \left((\Omega_k(\cdot))^2\right)^{1/2}$.

The estimated Morishima elasticity of substitution is the log derivative of the shadow price of inputs with respect to the log of the ratio of input quantities. Since the frontier of the input set is convex, this elasticity must be positive. Similar to Färe et al. (2010) and Chambers et al. (2013), the difference between the estimated and true elasticity equals

$$\mathrm{E}_k(\hat{\varphi}, \hat{\psi}, \hat{\chi}) = x_{1k}\left(\frac{\partial^2 \vec{D}_i(\bar{y}, x_{1k}, \tilde{x}_{2k}; g) / \partial x_1 \partial x_2}{\partial \vec{D}_i(\bar{y}, x_{1k}, \tilde{x}_{2k}; g) / \partial x_2} - \frac{\partial^2 \vec{D}_i(\bar{y}, x_{1k}, \tilde{x}_{2k}; g) / \partial x_1^2}{\partial \vec{D}_i(\bar{y}, x_{1k}, \tilde{x}_{2k}; g) / \partial x_1} + \frac{\partial^2 x_{2k} / \partial x_1^2}{\partial x_{2k} / \partial x_1} \right) \tag{15}$$

and

$$\mathrm{E}_k(\hat{\gamma}, \hat{\delta}, \hat{\omega}) = 1 - \frac{\partial^2 \ln D(\bar{y}, x_{1k}, \hat{x}_{2k}) / \partial \ln(x_1)^2}{\partial \ln D(\bar{y}, x_{1k}, \hat{x}_{2k}) / \partial \ln(x_1)} + \frac{\partial^2 \ln D(\bar{y}, x_{1k}, \hat{x}_{2k}) / \partial \ln(x_1) \partial \ln(x_2)}{\partial \ln D(\bar{y}, x_{1k}, \hat{x}_{2k}) / \partial \ln(x_2)} + x_{1k} \frac{\partial^2 x_{2k} / \partial x_1^2}{\partial x_{2k} / \partial x_1}. \tag{16}$$

Note that the true elasticity of substitution corresponding to the true log models is given by $1-\beta_1^L$ and that, as before, x_{2k} is either x_{2k}^P or x_{2k}^L. Our second criterion assesses the average difference between the true and estimated curvature of the frontier and is given by $K^{-1}\sum_k \left((\mathrm{E}_k(\cdot))^2\right)^{1/2}$.

We formulate four econometric specifications for every true technology and estimate them using the Aigner et al. (1977) approach. Three of them rely on the quadratic function and are defined by rotating the directional distance function's mapping vector from $g=(3,1)$ to $g=(1,1)$ and then to $g=(1,3)$—a setup allows us to see if the approximation quality depends on the direction of input contraction. The fourth specification relies on the translog Shephard's distance function that assumes a proportional reduction of both inputs. Approximation quality criteria that are based on the MRTS and elasticity are reported for each of the 108 cases in Tables 2 and 3, respectively. Figures 2 and 3 contain a selection of frontier estimates, obtained using input vectors $[x_1,\hat{x}_2]$ and $[x_1,\tilde{x}_2]$. Even though both functions can sometimes violate the global convexity of a true frontier, we chose not to impose any curvature conditions, since our main focus is the comparison of the functions' innate modeling properties.

We start by comparing the overall performance of the quadratic directional distance function and the translog Shephard's distance function. The MRTS-based discrepancies suggest that the translog functional form, whose corresponding benchmark quantities are reported in the bottom panel of Table 2, fares better than the quadratic function in approximately 85% (46 out of 54) of cases when the known technology is polynomial, and in about 56% (17 out of 27) of cases when it has a log configuration. In type-A models, where the translog frontier estimates often violate local convexity, the translog function outperforms the quadratic in all 27 cases. This is perhaps our most notable result, which contrasts sharply with the findings of Färe et al. (2010), who report that "… in the case of the true polynomial technologies, the quadratic function's global behavior is clearly superior to that of the translog function," as well as those of Chambers et al. (2013), who mention that "… the quadratic parameterizations are overall better than translog in approximating both types of true technologies…." However, as shown in Table 3, the advantage swings back in the quadratic function's favor when modeling the curvature. It dominates the translog function in 61% of cases when the true technology is polynomial and almost every time when it has a log structure.

Table 3 also suggests that while most of the quadratic specifications that assume $g = (1,3)$ beat the translog function, the latter can outperform the quadratic function when the mapping vector is rotated toward $g = (3,1)$. Consistent with the results of previous related studies, a quadratic specification whose mapping vector is most in line with the approach used to add inefficiency to the true models appears to dominate any of its quadratic counterparts. In other words, since this inefficiency component was simply added to the second input, it is the specification that assumes a predominantly southern direction of contraction that does the best job of tracking a true technology.

Another sign that the translog function may be better at approximating convex than concave frontiers is its large sample performance, which typically improves in true polynomial models, both when it is used to model the MRTS and the elasticity of substitution.[7] However, this result no longer holds in true log models, where the translog function's sample size related performance mostly deteriorates and whose corresponding translog frontier estimates usually

[7] This, too, differs from the results of Färe et al. (2010), who report precisely the opposite for concave frontiers.

violate global convexity of the true frontier (Figure 3). Sample size-related performance of the quadratic function is very good, but it depends on the directional vector in both types of true models. Table 2 suggests that as the number of observations increases, the MRTS-based benchmark decreases in 11 out of 18 cases when $g = (3,1)$, nearly always when $g = (1,1)$, and every time when $g = (1,3)$. Elasticity discrepancies associated with this functional form also decrease with an increase in sample size in the overwhelming majority of cases. The last two results are consistent with findings reported in existing simulation studies, suggesting that the quadratic function fares relatively well regardless of whether the known frontier is convex or concave.

Finally, the last set of findings that contradict the conclusions of our reference studies concerns the translog function's handling of an increase in the true curvature. Benchmark values in columns 2 and 4 (type-A models), as well as columns 8 and 10 (log models) of the bottom panel of Table 3 suggest that approximation quality usually improves when we add more curvature to a convex frontier. By contrast, the quadratic function always fares best when the true frontier is relatively flat.

3. Conclusion

Recent simulation studies by Färe et al. (2010) and Chambers et al. (2013) have compared the quadratic and translog functional forms and found that the former dominates the latter when used to approximate the concave frontier of the output set. Their authors have suggested that the key reason may be the translog function's inherent propensity to yield globally convex frontier estimates even when the true frontier is concave. We investigate this possibility by estimating a selection of convex frontiers of the input set and show that the translog function does behave much better. Although the quadratic function's overall performance remains adequate, its dominance of the translog form has diminished. For example, we found evidence that the translog form sometimes outperforms quadratic even when the true technology has a polynomial structure. The performance of either function can be rather uneven, and it depends on the characteristics of the known technology.

To put this analysis in a more general context, we note that our conclusions are consistent with the results of simulation studies by Wales (1977) and Guilkey et al. (1983), who have compared the performance of various functional forms, including the translog, but did not consider the quadratic function. Even though the translog form is clearly imperfect at modeling convex frontiers, it can sometimes outperform other functional forms, including those that are better than translog at approximating concave frontiers. Whenever possible, we recommend using both of these forms in empirical studies in order to model a true technology as best as possible.

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Table 1

Parameters Defining the True Technology

Polynomial Technologies

	P1	P2	P3
β_0^P	15.00	16.40	18.00
β_1^P	–2.80	–3.40	–3.85
β_2^P	0.15	0.22	0.25
β_3^P	-0.10×10^{-3}	-0.50×10^{-3}	0.10×10^{-4}
β_4^P	-0.20×10^{-3}	-0.15×10^{-3}	-0.22×10^{-3}
β_5^P	-0.10×10^{-5}	-0.20×10^{-4}	-0.10×10^{-4}

Logarithmic Technologies

	L1	L2	L3
β_0^L	0.91	1.17	1.44
β_1^L	–0.32	–0.57	–0.82

Table 2

Approximation Criterion Based on the Marginal Rate of Technical Substitution

Directional Input Distance Function

	P1A	P2A	P3A	P1B	P2B	P3B	L1	L2	L3
				g = (3, 1)					
K=50	0.691	0.620	0.586	0.736	0.708	0.706	2.786	2.049	2.278
K=100	0.642	0.583	0.553	0.731	0.716	0.651	3.135	2.154	2.233
K=500	0.674	0.607	0.574	0.717	0.691	0.651	3.471	2.010	2.060
				g = (1, 1)					
K=50	0.589	0.582	0.569	0.612	0.592	0.603	2.128	1.827	2.115
K=100	0.550	0.542	0.522	0.583	0.553	0.540	2.265	1.753	1.807
K=500	0.530	0.516	0.489	0.556	0.531	0.512	2.175	1.557	1.690
				g = (1, 3)					
K=50	0.467	0.578	0.453	0.440	0.465	0.453	1.245	1.304	1.466
K=100	0.370	0.411	0.369	0.362	0.370	0.370	1.199	1.168	1.296
K=500	0.310	0.338	0.305	0.314	0.301	0.303	1.009	0.955	1.094

Shephard's Input Distance Function

	P1A	P2A	P3A	P1B	P2B	P3B	L1	L2	L3
K=50	0.384	0.369	0.368	0.459	0.464	0.456	1.306	1.482	1.928
K=100	0.306	0.293	0.282	0.407	0.408	0.412	1.331	1.513	2.012
K=500	0.253	0.231	0.216	0.432	0.387	0.368	1.435	1.568	2.016

Table 3

Approximation Criterion Based on the Morishima Elasticity of Substitution

Directional Input Distance Function

	P1A	P2A	P3A	P1B	P2B	P3B	L1	L2	L3
					g = (3, 1)				
K=50	0.301	0.424	0.454	0.655	0.942	1.131	1.331	1.571	1.808
K=100	0.298	0.416	0.450	0.644	0.936	1.114	1.321	1.576	1.809
K=500	0.290	0.412	0.447	0.621	0.916	1.102	1.319	1.573	1.800
					g = (1, 1)				
K=50	0.206	0.315	0.361	0.460	0.756	0.913	1.325	1.560	1.774
K=100	0.195	0.306	0.349	0.450	0.716	0.882	1.295	1.553	1.728
K=500	0.184	0.299	0.344	0.423	0.688	0.869	1.260	1.528	1.716
					g = (1, 3)				
K=50	0.162	0.200	0.214	0.494	0.552	0.719	1.290	1.351	1.422
K=100	0.107	0.160	0.189	0.289	0.412	0.489	1.169	1.270	1.398
K=500	0.064	0.141	0.181	0.178	0.330	0.442	1.057	1.224	1.366

Shephard’s Input Distance Function

	P1A	P2A	P3A	P1B	P2B	P3B	L1	L2	L3
K=50	0.523	0.440	0.426	0.476	0.726	0.860	1.869	1.901	1.800
K=100	0.385	0.422	0.402	0.447	0.700	0.861	2.002	1.925	1.824
K=500	0.401	0.364	0.348	0.386	0.697	0.868	1.850	1.853	1.816

Figure 1

Frontiers of the Input Set Corresponding to the True Polynomial and Log Technologies

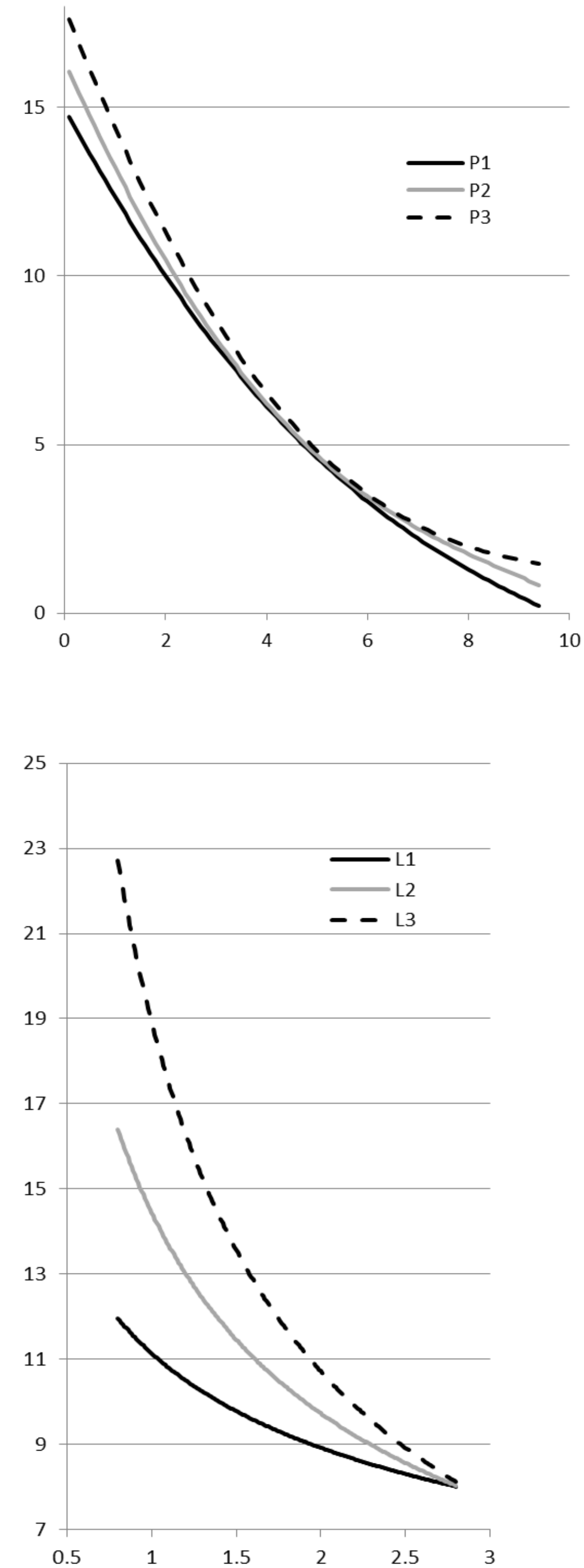
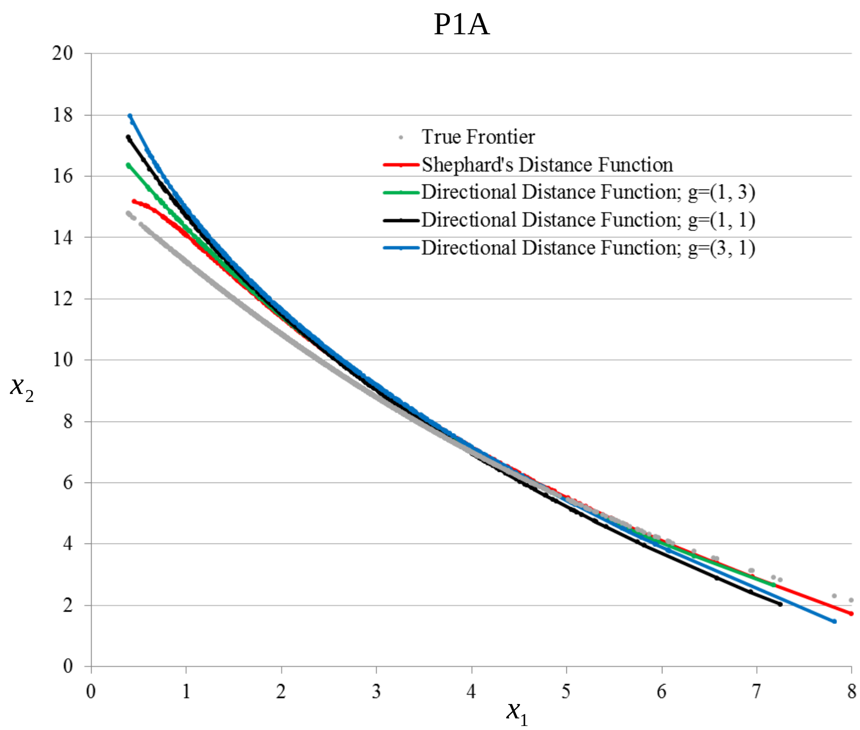


Figure 2

Frontier Estimates Corresponding to Selected Polynomial Technologies



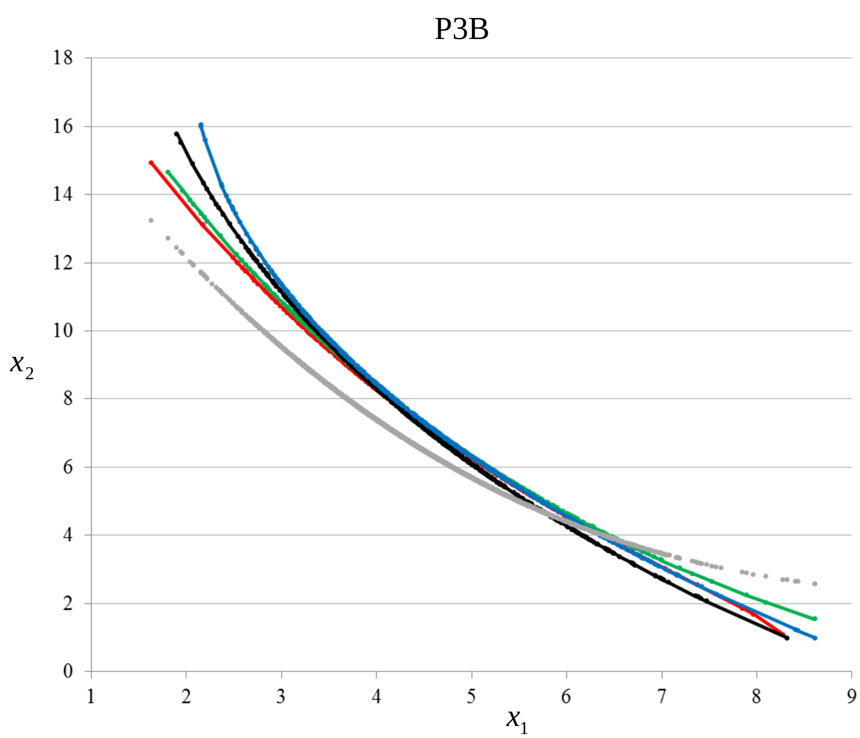


Figure 3
Frontier Estimates Corresponding to Selected Log Technologies