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Public versus Private Insurance System with (and without) Transaction Costs: Optimal Segmentation Policy of an Informed monopolist

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Abstract

Computer mediated transactions ([Varian, 2010]) allow insurance companies to customize their contracts while transaction costs limits this tendency toward customization. To capture this, we develop a complete-information framework in which it is costly to design a new market segment when the segmentation policy (number and design of segments) is endogenously chosen. Both the case of a private and a public insurer are considered. Without transaction cost, these two insurance systems are equivalent in terms of social welfare and participation. With transaction costs, this equivalence is not anymore true and the analysis of this difference is the subject of this article.

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1 Introduction

In his well-known article devoted to the organization of the (health) insurance market, [Diamond, 1992] reports that roughly 12% of the revenue of the U.S. health insurance industry goes to the administrative expenses. These expenses, called transaction costs in this article, represent all the *bureaucratic fixed costs* to run and manage an insurance company and do not by definition include the (expected) cost of claims. For instance, when the insurance company decides to propose a new insurance contract designed to a specific market segment, there are selling and marketing costs[1] that are examples of such transaction costs. While there is large body of academic literature in field of microeconomics of insurance (see for example the recent review of [Dionne et al., 2013]), only few papers explicitly analyze the case in which there are transaction (or production) costs. A recent review of this specific insurance literature can be found in [Ramsay and Oguledo, 2012], see also [Ramsay et al., 2013]. Following the seminal article of [Rothschild and Stiglitz, 1976], it is frequently assumed in the literature that the market segmentation is exogenous; there are typically two homogenous market segments and only one type of insurance system is considered, most often a private one. The aim of the present work is to bridge the gap in that we endogenize the market segmentation when there are transaction costs but we also consider a public and a private insurer.

We develop a framework in which the market segmentation is *endogenous* in that both the design and the number of market segments are chosen by the insurer when there is a positive set-up cost to design a new market segment. With the advances of the technology, big data, connected objects..., called more fundamentally *computer mediated transactions* by [Varian, 2010, Varian, 2014] (see also [Derez, 2016]), it is now possible to write terms in the insurance contract that were previously unobservable. A simple example of this is the possibility to use a GPS device, that is, a computer transmitter (say in the trunk of the car) that records among other things the vehicle's speed. As a result, this allows the insurance company to observe the driving style of a policy holder, and thus to offer a "pay how you drive contract". Computer mediated transactions, when possible, can be seen as a way to sharply limit adverse selection and moral hazard and thus facilitates personalization and customization of contracts[2].

[1][Rejda, 2014] offer an overview of the various (practical) methods for selling and marketing insurance products for different types of insurance, health, life, property and casualty.

[2]It is frequently noted that with big data, the adverse selection could indeed be reversed ([Siegelman, 2014], i.e., the insurer would be able to know more about the risk than the agent herself.

In a world with complete information but without transaction costs, the two insurance systems, public and private, turns out to be equivalent in terms of social welfare and participation. The private (profit-maximizer) insurer offers each (potential) policyholder a contract in which the premium is equal to her reservation price (i.e., each of agent is left without any surplus[3]) while the public (social welfare-maximizer) insurer offers each (potential) policyholder a contract in which the premium is equal to her expected loss (i.e., the expected profit is equal to zero). All the agents are insured under each system (i.e., full participation) and the social welfare is identical. With transaction costs, for instance when there is a positive set-up cost to design a new market segment, perfect customization is not anymore possible and the analysis of its consequences is the subject of the present work.

We consider a single (public/private) insurer in complete information–the perfectly informed monopolist–that faces a continuum of agents (or types) and we build on the idea that it is costly to design a new market segment. In our framework, the segmentation of the policy holders into risk groups (or market segments) occurs not because of the informational asymmetry, as in most models (see e.g., [Bossert and Fleurbaey, 2002]), but because of the existence of transaction costs that increase with the number of market segments (or groups). It is important to point out at this stage that our purpose is not to claim that informational asymmetry is not an important issue for an insurance company[4]. It is rather to say that market segmentation is also an important issue in the insurance industry that have not been, to the best of our knowledge, explicitly studied. The framework developed here should thus be thought of as a complement of the classical insurance models rather than a substitute. Throughout this article, a *segmentation policy* is defined by the choice of the number of groups, their design and the premiums.

As usual in Economics, the private insurer is assumed to choose the segmentation policy to maximize its expected profits. The optimization problem turns out to be a simple example of a mixed-integer problem (MIP) as the design of the market segments is a continuous problem while the choice of the number of segments is a discrete one. An important feature of the optimal segmentation policy chosen by the private insurer is that it is never optimal to insure the riskiest agents. Although there are many possible social objective functions,

See e.g., [Villeneuve, 2000] for such a model although his argument is not related to big data.

[3]Such a situation is generally called perfect (or first-degree) price discrimination in Economics

[4]For instance, with genetic testing, an insurer may have the possibility to be completely informed about the type of a potential policy holder. However, such genetic testing is not allowed in many jurisdictions, see the Oviedo conventions in Europe http://www.coe.int/en/web/bioethics/oviedo-convention. An interesting discussion of genetic testing and insurance is provided in [Durnin et al., 2012].

following [Brown and Sibley, 1986, Steinberg and Weisbrod, 2005] among others, the public insurer is assumed to choose the segmentation policy to maximize the total policyholders' surplus under a budget constraint of zero expected total profit. We show that as long as the constraint of zero expected profit is equal to zero, full participation is socially optimal. However, this fairly general result gives no clue regarding the choice of the segmentation policy and/or the desirability of cross-subsidization. We thus explicitly consider two different ways of choosing the segmentation policy, one without cross-subsidization, i.e., such that the expected profit of *each* group is equal to zero, and one with cross-subsidization, i.e., only the (aggregate) budget constraint is satisfied. In both case, we make the assumption that the riskiest agents are proposed a contract. Without cross-subsidization, the algorithm used to design the groups is simple; the public insurers first tries to design a group with the riskiest agents. If it is possible, the public insurer then tries to design a second group and so on and so forth as long as this is possible. When there are transaction costs, only a finite number of groups can be designed. Of course, if transaction costs are too high, no group can be designed. We then explicitly analyze the case with cross-subsidization, which leads to more complex problem as there are many ways to introduce such cross-subsidization. We shall assume that the public insurer behaves as a profit maximizer and then uses the entire profit realized to create a new, subsidized, subset of insured agents. Quite interestingly, the two approaches (with an without cross-subsidization) may generate a trade-off between the social welfare (i.e., surplus) and the level of participation (i.e., the percentage of agents that are insured). As the segmentation problem turns out to be a mixed-integer problem for the insurer (public or private), we consider the particular case of our model (square root utility function/uniform distribution...) that allows us to offer a numerical analysis. The analysis of this simple model interestingly shows that the desirability of the public insurance system increases with transaction costs. When transaction costs are "high", the social welfare but also the level of participation under a public system of insurance are much higher than under a private insurance system. In such a high transaction costs case, we also show that the public insurer faces a trade-off between social welfare and (the level of) participation.

The paper is organized as follows. Section two of this paper is devoted to the framework. Section three and our are devoted respectively to the benchmark (i.e., no transaction costs) and the case in which there are positive transaction costs. Section five is devoted to the numerical analysis through is a simple particular case of our fairly general framework. Section six is devoted to the conclusion and possible extensions. All the proofs are relegated in appendix A.

2 Framework

We present in this section the main assumptions of our set-up. Let θ denotes the probability of damage (type) but also the index of the agent. Throughout this paper, an agent will be labeled using her type θ. The underlying damage may be the income loss that results from a car accident, a fire, a theft... we don't specialize the model to a specific category of insurance.

Assumption 1 *The type θ is distributed in $\Theta := [0, 1]$ according to some regular distribution function F that admits a density f.*

Since the set of types is the unit compact of the set of reals, assumption 1 means that we consider a *continuum* of agents. In practice, such a continuum of agents makes no sense. However, working with a set of agents that has the cardinality of the continuum nicely captures the idea of a large economy. From the point of view of an insurer, this means when considering a subset of agents called a market segment (or simply a group), defined by the interval $[a, b]$ (with $a > 0$ and $b < 1$), a weak law of large number is shown to hold. As we shall see, working with a continuum also makes the segmentation model more tractable as it reduces to a continuous problem. Regarding now the regularity of the function F, we shall assume F to be twice continuously-differentiable and strictly increasing on $(0, 1)$. As a result, the density $f(\theta)$ is continuously-differentiable and strictly positive on $(0, 1)$. Moreover as usual, each agent θ will be supposed to be an expected utility maximizer and the utility function U is assumed to be a positive, twice continuously differentiable (strictly) increasing and (strictly) concave function of the wealth. Let $W > 0$ be the initial (finite) wealth of each agent, and let $L \in (0, W]$ be the loss of each agent in case of damage. As in [Stiglitz, 1977, Chade and Schlee, 2014], agents are differentiated by the probability of damage only. Although this framework is probably the simplest one from an economic point of view, it is yet rich enough to analyze various topics in the microeconomics of insurance. Moreover, as we shall see, cross-subsidization (for a public insurer) still makes sense although each agent is endowed with the same wealth. The final wealth of each agent is the random variable that takes two values only; $W - L$ with probability θ and W with the complementary probability $1 - \theta$ so that the expected utility (of the final wealth) of agent θ thus is equal to

$$v(\theta) = \theta U(W - L) + (1 - \theta)U(W)) \tag{1}$$

and is a decreasing function of θ. In our model, each agent may have the possibility to be insured against the loss due to the damage.

Assumption 2 *Information is complete, i.e., the utility function, the type of each agent and the distribution function of the types are known from the insurer.*

In most models, the insurance company faces an adverse selection problem in that it is unable to make a discrimination between the agents since the probability of damage (i.e., the type) is assumed to be unknown. Only the distribution function of the type is known. We argue here that with the development of new technologies and the possibility to use powerful computers, it makes now sense to consider models in which the insurer has (or may have) much more information about the types of the agents than few decades ago. Consider for instance the case of automobile insurance. It is well-known that age and sex[5] are two important factors explaining the probabilities of accidents (e.g., [Dionne et al., 2013]). But with the advances of the technology, it is now possible for the auto insurance industry to offer "telematics-based" insurance products. By using a GPS devices with integrated accelerometers installed in the car (i.e., satellite sensors), this mechanism will transmit a huge number of information such as date, location, speed etc... and the premium can be based on these information. As a result, instead of paying a premium based on an average probability (of an underlying group of agents), an agent who has chosen such a pay-how-you-drive contract will pay a premium based on her *own type*. More generally, with computer mediated transactions (see [Varian, 2010], see also [Varian, 2014]), it is now possible introduce terms in the contracts that were previously unobservable.

In practice, the segmentation of customers is obviously chosen by the insurance company, possibly with the help of statistical methods (e.g., [Samson, 1986, Smith et al., 2000, Yeo et al., 2002]. It is common for insurers to group heterogeneous policy-holders into different *risk groups* (also called class rating, market segments, groups...) based on a set of observable factors such as age, gender, occupation etc... The segmentation process is a key activity for the insurance company as a poor market segmentation may generate important losses. In a complete information model, the segmentation of the market is direct in that each agent is assigned by the insurer to a group as a function of her type, i.e., each agent is proposed a single insurance contract. This contrasts with incomplete information models à la Rothschild and Stiglitz in which each agent is proposed a menu of contracts where the final choice of the contract is left to the agent.

[5] Since 2012, in the European Union, insurance companies have to charge the same price to men and women for the same insurance products, without distinction on the grounds of sex.

Assumption 3 *The design of a new contract (i.e., a new group of policyholders) generates a constant transaction cost equal to K.*

In theory, transaction costs represent *all* the administrative fixed expenses required to design, process and administrate a new contract proposed by the insurer to a group of agents. For concreteness, following [Allard et al., 1997, Liu and Browne, 2007] (see also [Ramsay and Oguledo, 2012]), we assume that K is the (per-contract) fixed cost of designing and marketing an insurance contract, i.e., K can be thought of as a set-up cost[6]. As a result, if $n \geq 1$ market segments have been designed by the insurer, the total transaction cost is equal to

$$K_n = nK \tag{2}$$

Equation (2) can be thought of as the simplest transaction cost function such that when n tends to infinity, K_n also tends to infinity. As a result, it makes only sense for an insurer to choose a *finite* number $n \geq 0$ of market segments, possibly zero if K is too high. However, the real difficult task for the insurer is the design of the groups. In the particular case in which there is no set-up cost, i.e., $K = 0$, this task is easy as a personalized contract $C(\theta) = (P(\theta), I(\theta))$, based solely on her type, is proposed to *each* agent θ. When $K > 0$, the insurer public or private can not anymore offer such a customized contract to each agent. Instead, the set of agents has to be divided into a number $n \geq 0$ of groups and each agent assigned to the same group is proposed the same insurance contract.

Assume that the insurer has decided to set $n \geq 1$ groups or market segments and let $G_i := [\theta_i, \theta_{i+1}) \subseteq [0, 1]$ be the market segment i, defined as the right-open interval $G_i := [\theta_i, \theta_{i+1})$ for $i = 1, ..., n$ such that $G_i \bigcap G_j = \emptyset$ for $i \neq j$, i.e., each agent is assigned to one group only. By definition of a market segment, each agent θ assigned to the group G_i by the insurer is proposed the same *non-mandatory* insurance contract $C_i = (P_i, I_i)$ on a take it or leave it basis. The premium P_i that each agent has to pay to be insured and $I_i \leq L$ is the indemnity paid by the insurer in case of damage. To be accepted by an agent θ assigned to the group G_i, the so-called *participation constraint* must thus be satisfied. As we work with a complete information model, there is thus no incentive constraint. Moreover, in complete information, the choice of the design of the group G_i, i.e., θ_i, θ_{i+1} and the choice

[6] Note interestingly that contrary to most goods, an insurance contract is *nominative*, and thus can not be subject to parallel trade, i.e., those who pay a low premium can not resell their contract to those who pay a high premium. As a result, the costs associated to the prevention and the mitigation of parallel trade, which may be important for goods such as pharmaceuticals, books, computers, cars (see e.g., [Braouezec, 2012]) are nonexistent for insurance contracts.

of the premium P_i must be done in a consistent way. If not, say if only the subgroup $[\widetilde{\theta}_i, \theta_{i+1})$ (with $\widetilde{\theta}_i > \theta_i$) accept the contract for which the premium is equal to P_i, then, P_i is inconsistent with θ_i since θ_i is simply irrelevant. When the choices are consistent, all the agents of the group G_i should thus accept the contract. As in [Johnson and Myatt, 2003, Malueg and Schwartz, 1994, Oren et al., 1984] among others in a non-insurance framework, we shall consider the case of adjacent intervals.

Assumption 4 *For each $n \geq 1$, a market segmentation is defined as a collection of adjacent intervals $G_1, ..., G_n$, where $G_i := [\theta_i, \theta_{i+1})$ and $\theta_{n+1} \leq 1$.*

In such a case, the family $(G_i)_{i=1}^n$ forms a partition of the compact subset $[\theta_1, \theta_{n+1}] \subseteq [0,1]$. This property of adjacent intervals is actually a convexity property. If agents $\overline{\theta}$ and $\underline{\theta}$, with $0 \leq \underline{\theta} < \overline{\theta} \leq 1$, are proposed a contract (which is accepted), possibly not the same, then, for any $\alpha \in (0,1)$, agent $\theta = \alpha\overline{\theta} + (1-\alpha)\underline{\theta}$ is proposed an insurance contract. For a public insurer, it would be strange to choose not to insure an intermediate category of agents. Considering a market segmentation as a collection of adjacent market segments can be thought of as a natural underwriting principle. The situation is more delicate for the private insurer as the optimal market segmentation needs not be such that *all* the market segments are adjacent intervals. In appendix B, we show that a typical situation in which some market segments are possibly non-adjacent is when the density is bimodal. We also discuss the optimal market segmentation when groups are allowed to be non-adjacent[7]. Beyond the reduction of dimensionality of the problem, the main interest of this assumption is that it allows us to easily compute, for each $n \geq 1$, the level of participation, i.e., the fraction of agents that are insured. Let

$$C([0,1]^{n+1}) := \{(\theta_1, ..., \theta_{n+1}) \in [0,1]^{n+1} : 0 \leq \theta_i \leq \theta_{i+1},\ i = 1, 2, ..., n, \text{with } \theta_{n+1} \leq 1\} \quad (3)$$

be the set of market segmentation, and note that $C([0,1]^{n+1})$ is a compact and convex of subset of $[0,1]^{n+1}$. For notational simplicity, and when there is no confusion, we shall write θ_i instead of $\theta_i^{(n+1)}$, and we shall talk about groups or market segments rather than intervals. Let $\vec{\theta}_{(n+1)} = (\theta_1, ..., \theta_{n+1}) \in C([0,1]^{n+1})$ be a given market segmentation.

Definition 1 *A segmentation policy is defined by the choice of a number $n \geq 1$ of groups (or contracts), a market segmentation $\vec{\theta}_{(n+1)}$ and a set of premiums $P_1, ..., P_n$.*

[7] In the last section of this article, we offer a numerical analysis based on a specific model in which θ is assumed to be uniformly distributed in $[0,1]$, and it is shown in appendix B that the optimal market segmentation must form a collection of adjacent market segments.

As usual in Economics, the public insurer is assumed to choose the segmentation policy in order to maximize the total surplus subject to a budget constraint of no expected profits (e.g., [Brown and Sibley, 1986, Steinberg and Weisbrod, 2005]) while the private insurer is assumed to choose the segmentation policy in order to maximize its total expected profits. Under a private system of insurance, the riskiest agents typically remain uninsured in that they are not proposed any contract. In such a situation, as recalled in [Rejda, 2014, chap 8], a government insurance program might be necessary to insure those agents. In this paper, with or without cross-subsidization, the riskiest agents, i.e., a group of the form $[\widetilde{\theta}, 1]$, will be proposed a contract from the public insurer as long as the budget constraint can be satisfied. For this to make economic sense, we make the implicit assumption, as usual in pure adverse selection models, that the type θ is *exogenous* to the agent, i.e., it is not influenced by her own decision. Otherwise, it would be economically why unclear why these riskiest agents should be always insured.

3 No transaction cost : The benchmark

We call benchmark the situation in which there is no transaction cost (i.e., $K = 0$) since, as we shall see, the two insurance systems are equivalent in terms of efficiency, i.e., social welfare and participation, i.e., percentage of agents that are insured. This case is obviously particular since it is possible for the insurer to *costlessly* design a customized insurance contract to each agent, that is, to propose a contract $C(\theta) = [P(\theta); L(\theta)]$ based on a take or leave it basis that explicitly depends on the type θ of the agent. In this case, a group is simply the singleton $\{\theta\}$. As we shall now see, the optimal contract, whether the insurer is public or private, is such that $L(\theta) = L$ for each $\theta \in [0, 1]$, i.e., the optimal contract is a full coverage one.

Consider a given agent θ and assume she has been offered a full coverage insurance contract with a premium equal to P. Since the insurance contract is not mandatory, depending on the premium P, the agent θ may refuse it. Let $\overline{P}(\theta)$ be her willingness to pay, that is the *maximum premium* the agent θ is ready to pay for the full coverage insurance contract that pays L in case of damage. By definition, $\overline{P}(\theta)$ is such that $U(W - \overline{P}(\theta)) = v(\theta)$ so that

$$\overline{P}(\theta) = W - U^{-1}(v(\theta)), \quad \theta \in [0, 1] \tag{4}$$

For a full coverage insurance contract, a *closed formula* can be obtained for the willingness-to-pay. This is indeed not the case for a partial coverage insurance contract since the willingness-to-pay can only be defined implicitly (see equation (62) in appendix B).

Lemma 1 *For all $\theta \in (0,1)$ $\overline{P}(\theta) > \theta L$ and such that $\overline{P}(0) = 0$, $\overline{P}(1) = L$. Moreover, $\overline{P}(\theta)$ is an increasing and concave function of θ.*

From risk-aversion, an agent $\theta \in (0,1)$ is always ready to pay more than her expected loss, i.e., $\overline{P}(\theta) > \theta L$. The two extreme agents, i.e., $\theta = 0$ and $\theta = 1$ are special in that the first one never find useful to be insured while the other one is willing to pay her entire wealth.

Private insurer. As is well-known from Stiglitz (1977) (see also proposition 1 in the review paper of Dionne et al. 2013), the profit-maximizing menu in such a case is simple. The optimal indemnity $I^*(\theta)$ is equal to L for each θ so that each agent is *fully insured*, and the optimal premium to be charged to agent θ, $P^*_{prv}(\theta)$, is equal to $\overline{P}(\theta)$), her maximum willingness to pay for the full insurance contract (see equation (4)). As a result, while each agent is insured, she is left without any surplus. For a given agent θ, the welfare, profit and surplus, thus is equal to the expected profit $W - U^{-1}(v(\theta)) - \theta L$, that is, the premium paid by agent θ equal to $P^*_{prv}(\theta) = W - U^{-1}(v(\theta)$ minus the expected cost of claims equal to θL.

Public insurer. Since a risk-averse agent will always decide to be fully insured as long as the premium is actuarially fair, the public insurer can offer to each agent θ a full coverage insurance contract such that the premium $P^*_{pub}(\theta)$ is equal to her expected loss θL. As a result, the expected profit is equal to zero so that the welfare is equal to the surplus of agent θ, that is $\overline{P}(\theta) - P^*_{pub}(\theta) = W - U^{-1}(v(\theta)) - \theta L$.

From the above discussion, whether the insurer is public or private, for each agent θ, the welfare is invariably equal to $W - U^{-1}(v(\theta)) - \theta L$. Let Ψ^*_{prv} and Ψ^*_{pub} be the social welfare under the private and the public insurance system and note that Ψ^*_{prv} is equal to the (maximum) total expected profits of the private insurer. The following proposition summarizes the above discussion.

Proposition 1 *When there are no transaction costs, under both insurance systems, public and private*

1. *all the agents are insured, that is, there is no exclusion.*

2. *the social welfare is equal to*

$$\Psi^*_{prv} = \Psi^*_{pub} = W - \int_0^1 U^{-1}(v(\theta))f(\theta)d\theta - L\,\overline{\theta} \tag{5}$$

where $\overline{\theta} := \int_0^1 \theta f(\theta)d\theta$ is the average probability based on the overall set of agents.

The no-transaction cost case is particular since under both insurance system, private and public

1. The social welfare is identical.
2. Each agent is offered a full coverage insurance contract.
3. A *continuum* of contracts is offered.

The first property means that there is *no loss of efficiency* since under both systems, the social welfare is identical. If one considers the social welfare as the unique measure to evaluate the "quality" of the market, then, the two insurance systems should be equivalent. The second property says that the contract proposed by the insurer (public or private) to each agent is such that the indemnity in case of damage is equal to the loss. Finally, the last property says that since a customized contract is proposed to each agent, a *continuum* of contracts is offered since there is a continuum of agents. Of course, in real insurance markets, this property makes no sense. The number of contracts (or groups) is always finite and much smaller than the number of policyholders. Within our framework, it suffices to introduce a set-up cost per contract, arbitrarily small, to obtain this finiteness property. However, exactly how the insurer, public or private will design the groups and what will be the consequences on efficiency is the subject of this paper.

4 Positive transaction costs

An important feature of the benchmark is that a full coverage contract is proposed to each agent whether the insurer is public or private. With transaction costs, we shall also consider *full coverage* insurance contracts whether the insurer is public or private. Under this assumption, it becomes easier to compare the two cases, i.e., with and without transaction costs. In any event, in appendix B, partial insurance contracts are discussed.

Consider a given agent θ and assume that she has been assigned to the group G_i. All the agents assigned to that group are proposed the uniform contract $C_i := (P_i, L)$ on a take-it-or-leave-it basis, where P_i is the premium each agent of G_i has to pay. As usual, it is assumed that agent θ accepts the contract if $\overline{P}(\theta) \geq P_i$ and refuses it otherwise. The surplus of agent θ who is offered the contract C_i thus is equal to $CS(\theta) = \max\{0, \overline{P}(\theta) - P_i\}$. Since the choice of each group G_i is assumed to be consistent with the choice of the premium, as we have seen, this means that $P_i \leq \overline{P}(\theta_i)$ so that all the agents of that group accept the contract. The surplus of policyholders assigned to the group G_i thus is equal to

$$CS_i = \int_{\theta_i}^{\theta_{i+1}} CS(\theta))f(\theta)d\theta = [F(\theta_{i+1}) - F(\theta_i)](W - P_i) - \int_{\theta_i}^{\theta_{i+1}} U^{-1}(v(\theta))f(\theta)d\theta \tag{6}$$

Let $A(\theta_i, \theta_{i+1})$ be the *average probability* (i.e., it is a conditional expectation) of the group $G_i = [\theta_i, \theta_{i+1})$ defined as

$$A(\theta_i, \theta_{i+1}) := \frac{\displaystyle\int_{\theta_i}^{\theta_{i+1}} \theta f(\theta)d\theta}{F(\theta_{i+1}) - F(\theta_i)} \tag{7}$$

and note that this average probability is such that $\theta_i < A(\theta_i, \theta_{i+1}) < \theta_{i+1}$. It is not difficult to show that $A(\theta_i, \theta_{i+1})$ is an *increasing function* of θ_i and θ_{i+1}. Let

$$\mathbb{E}(R_i) = \int_{\theta_i}^{\theta_{i+1}} P_i f(\theta)d\theta - L\int_{\theta_i}^{\theta_{i+1}} \theta f(\theta)d\theta = [P_i - A(\theta_i, \theta_{i+1})L]\,[F(\theta_{i+1}) - F(\theta_i)] \tag{8}$$

defines the (expected) *gross profit* of the group i and note that the second term in the rhs of equation (8) simply follows from equation (7). With such a formulation, equation (8) can interestingly be interpreted in terms of price, average cost and quantity. The premium P_i is the price paid, $A(\theta_i, \theta_{i+1})L$ is the average cost of claims and finally $[F(\theta_{i+1}) - F(\theta_i)]$ is the quantity sold. We call (expected) *net profit* of the group G_i the quantity defined as $\mathbb{E}(R_i) - K$, that is, the difference between the (expected) gross profit $\mathbb{E}(R_i)$, and the set-up cost K. For a given market segmentation, using equations (7) and (8), it is not difficult to show that the total (expected) net profit of the insurer is equal to

$$\mathbb{E}\Pi(\vec{\theta}_{(n+1)}) := \mathbb{E}R(\vec{\theta}_{(n+1)}) - nK = \underbrace{\sum_{i=1}^{n} P_i[F(\theta_{i+1}) - F(\theta_i)] - L\int_{\theta_1}^{\theta_{n+1}} \theta f(\theta)d\theta}_{\mathbb{E}R(\vec{\theta}_{(n+1)})} - nK \tag{9}$$

where $\mathbb{E}R(\vec{\theta}_{(n+1)}) = \sum_{i=1}^{n} \mathbb{E}(R_i)$ is the expected total gross profit.

At this stage, since each group $G_i = [\theta_i, \theta_{i+1})$ has the cardinality of the continuum, it is fairly intuitive to think that a law of large numbers should hold. Let $X(\theta)$ be the loss of agent $\theta \in G_i$ be a Bernoulli random variable equal to L with probability θ and zero with the complementary probability. The aggregate loss of the group G_i is defined as the following non-denumerable sum of independent random variables

$$X = \int_{G_i} X(\theta)dF(\theta) = \int_{G_i} X(\theta)f(\theta)d\theta \tag{10}$$

We know from [Judd, 1985] that we can not expect to prove a strong law of large numbers when one considers a continuum of independent random variables. However, as shown by [Uhlig, 1996], a weak law of large numbers can be shown to hold. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random variables. [Uhlig, 1996] proves that for a continuum of IID random variables (he actually assumes no correlation), one can obtain a convergence in mean-square, that is, when $n \to \infty$, $\mathbb{E}(X_n - X)^2) \to 0$. The result of [Uhlig, 1996] can unfortunately not be directly applied here as we work with a continuum of independent random variables that are *not* identically distributed. However, it is easy to obtain a weaker result, namely that when $n \to \infty$, $\mathbb{E}|X_n - X|) \to 0$ where $X_n = \sum_{j=1}^{n} X(\theta_j)f(\theta_j)(\theta_j - \theta_{j-1})$, with $\theta_0 = \theta_i$ and $\theta_n = \theta_{i+1}$ (as usual, when n tends to infinity, $\max(\theta_j - \theta_{j-1})$ must converges to zero). From the hierarchy of convergence, if X_n converges to X in mean, this implies that X_n converges to X in probability.

4.1 Optimal segmentation policy of the private insurer

Consider a given group of agents $G_i = [\theta_i, \theta_{i+1})$. From equation (8), as long as $P_i \leq \overline{P}(\theta_i)$, the total quantity $F(\theta_{i+1}) - F(\theta_i)$ but also the average probability $A(\theta_i, \theta_{i+1})$ remain identical. The expected gross profit of the group G_i thus is an increasing function of the premium $P_i \in [0, \overline{P}(\theta_i]$. As a result, it is optimal for a profit-maximizer insurer to set $P_i = \overline{P}(\theta_i)$ for each $i = 1, ..., n$. The segmentation policy thus reduces to the choice of $\vec{\theta}_{(n+1)} := (\theta_1, ..., \theta_{n+1})$, the market segmentation, and $n \in \mathbb{N}$, the number of market segments. Formally, the optimization problem of the private insurer is given by

$$\max_{(n;\vec{\theta}_{(n+1)})\in\mathbb{N}\times C([0;1]^{n+1})} \mathbb{E}\Pi(\vec{\theta}_{(n+1)}) := \sum_{i=1}^{n} \overline{P}(\theta_i)[F(\theta_{i+1}) - F(\theta_i)] - L\int_{\theta_1}^{\theta_{n+1}} \theta f(\theta)d\theta - nK \quad (11)$$

and is a simple example of a mixed-integer programming problem (MIP) because the choice of the market segmentation is a *continuous* problem while the choice of the number of groups is an *integer* problem. Given equation (11), it is clear that the optimal number of market segments is necessarily finite, possibly zero. Let

$$K_{prv} = \sup\{K \geq 0 : n^*_{prv}(K) = 1\} \quad (12)$$

and note that K_{prv} is such that $\mathbb{E}\Pi(\theta^*_1, \theta^*_2) = 0$. Let $n^*_{prv}(K) := n^* \in \mathbb{N}$ be the optimal number of market segment and $\vec{\theta}^*_{(n^*+1)} = (\theta^*_1, ..., \theta^*_{n+1})$ be the optimal market segmentation.

Proposition 2 *Assume that $K \leq K_{prv}$. The optimal segmentation policy $(n^*, \vec{\theta}^*_{(n^*+1)}) \in \mathbb{N}^* \times C([0;1]^{n^*+1}$ that solves the MIP defined by equation (11) is such that*

1. $\theta_1^* > 0$ *and* $\theta_{n+1}^* < 1$ *for any choice of* $n \geq 1$.

2. $n^* \to \infty$ *when* $K \to 0$

Note that the following proposition would hold under a more general transaction cost function, e.g., if K_n is a piece-wise constant function such that $\lim_{n\to\infty} K_n \to \infty$. Part 1 of the above proposition says that under the optimal segmentation policy, as long as K is positive, agents of the group $[0, \theta_1^*]$ and agents of the group $[\theta_{n^*}^*, 1]$ are uninsured. Only safest and the riskiest agents are not proposed any insurance contract. Part 2 says that, as expected, when K goes to zero, the optimal number of market segments tends to infinity. In this limiting case, almost 100% of the agents are proposed an insurance contract.

4.2 Optimal segmentation policy of the public insurer

As already said, the segmentation policy of the public insurer is chosen to maximize the social surplus under a budget constraint that the (expected) total net profit is equal to zero. It will be convenient to express the premium P_i as a function of the average probability of the group G_i. Let the premium of the group G_i be defined as

$$P_i = A(\theta_i, \theta_{i+1})L_i \leq \overline{P}(\theta_i) \tag{13}$$

where L_i is a positive number chosen by the insurer. For a given number $n \geq 1$ of groups, since $P_i \leq \overline{P}(\theta_i)$ for each $i = 1, ..., n$, from equation (8), the expected gross profit of the insurer is equal to

$$\mathbb{E}(R_i) = (L_i - L)A(\theta_i, \theta_{i+1})[F(\theta_{i+1}) - F(\theta_i)] \tag{14}$$

and is non-negative as long as $L_i \geq L$. On can also define the quantity L_i as follows $L_i = (1 + \lambda_i)L$ so that the parameter λ_i can now be interpreted as the *loading factor* of the group G_i. The expected profit thus is positive as long as the loading factor is positive. For a given $n \geq 1$, let $\vec{\theta}_{(n+1)} = (\theta_1, ..., \theta_{n+1})$ be a given market segmentation and note that the total expected net profit is equal to

$$\mathbb{E}\Pi(\vec{\theta}_{(n+1)}) = \sum_{i=1}^{n}(L_i - L)A(\theta_i, \theta_{i+1})[F(\theta_{i+1}) - F(\theta_i)] - nK \tag{15}$$

Proposition 3 *Let* $n \geq 1$ *and assume that the expected total net profit of the public insurer given by equation (15) is equal to zero. Then, for any choice of* $n \geq 1$*, the total policyholders' surplus reduces to* $\Psi(\theta_1, \theta_{n+1})$ *and its maximization leads to* $\theta_1^{**} = 0$ *and* $\theta_{n+1}^{**} = 1$*, that is, full participation is socially optimal.*

Proposition 3 is a fairly general result as it depends only on risk-aversion. When at least one market segment is designed, it simply says that *all* the agents should be insured regardless of transaction costs. However, it gives no clue regarding the choice of the segmentation policy and/or the desirability of cross-subsidization since the policyholders' surplus turns out to be a telescopic sum that depends only on θ_1 and θ_{n+1}. Nothing can thus be said on $\theta_2, ..., \theta_n$. Note importantly that proposition 3 does not rule out cross-subsidization because the constraint only states that the *total* (expected) net profit has to be equal to zero. It thus allows the public insurer to make some positive profit with one (or more than one) market segment as long as this profit is (entirely) used to subsidize another group of agents. We shall consider two ways of determining the optimal segmentation policy of the public insurer; with and without cross-subsidization. In both cases, we make the following addtional assumption. For each $n \geq 1$, the public insurer sets

$$\theta_{n+1} = 1 \tag{16}$$

The above assumption means that for any choice of $n \geq 1$, the public insurer always decide to offer a contract to the riskiest agents. If only one group can be designed, i.e., $n = 1$, this group is of the form $G_1 = [\theta_1, 1]$ for some $\theta_1 < 1$. This assumption thus gives a *priority* to the riskiest agents, which can indeed be thought of as an underwriting principle. This choice contrasts with the private insurer since it is *never optimal* to offer a contract to the riskiest agents from (see proposition 2), i.e., for a private insurer, it is always the case that $\theta_{n+1} < 1$ for each $n \geq 1$.

No cross-subsidization. By construction, the premium of the group G_i is equal to $P_i = A(\theta_i, \theta_{i+1})L_i \leq \overline{P}(\theta_i)$. Since there is no cross-subsidization, the quantity L_i must be chosen such that the expected profit of the group i must be equal to zero. No cross-subsidization thus is equivalent to

$$\mathbb{E}(R_i) = K, \quad \text{for each} \;\; i = 1, ..., n \tag{17}$$

Using equation (14), it is easy to show that $\mathbb{E}(R_i) = K$ is equivalent to $L_i = L + \frac{K}{[F(\theta_{i+1}) - F(\theta_i)]A(\theta_i, \theta_{i+1})}$. Inserting L_i in equation (13), the premium charged to each agent of the group G_i thus is equal to

$$P(\theta_i, \theta_{i+1}, K) := P_i = A(\theta_i, \theta_{i+1})L + \frac{K}{[F(\theta_{i+1}) - F(\theta_i)]} \qquad i = 1, ..., n \tag{18}$$

Fact 1 *Assuming that $\theta_{i+1} \in (0, 1)$ is given, the premium $P_i = P(\theta_i, \theta_{i+1}, K)$ is an increasing function of θ_i such that $P(\theta_i, \theta_{i+1}, K)$ tends to infinity when θ_i tends to θ_{i+1}, for $i = 1, ..., n$.*

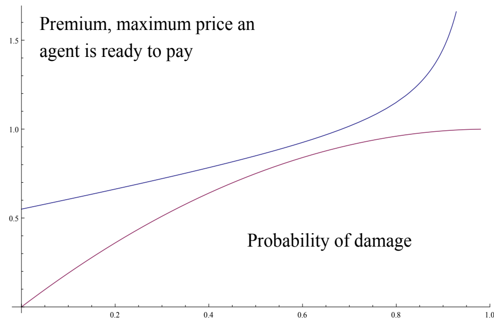

Figure 1: No solution

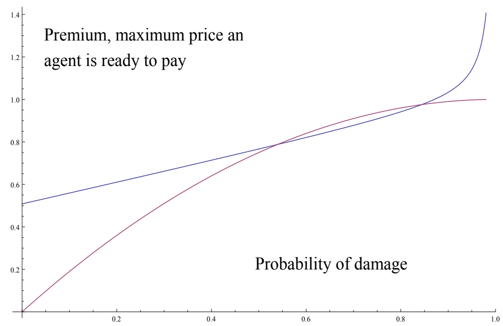

Figure 2: Multiple solutions

The premium given by equation (18) is interesting to look at since it is the sum of two types of *average costs*. The first one, equal to $A(\theta_i, \theta_{i+1})L$, reflects the expected cost of claims of agents assigned to the group G_i. The second one, equal to $\frac{K}{[F(\theta_{i+1})-F(\theta_i)]}$, reflects the average transaction cost of that group G_i and is a decreasing function of the size the of group. If we use the loading factor λ_i, the premium of the group is equal to $P_i = (1+\lambda_i)A(\theta_i, \theta_{i+1})L$ so that $\lambda_i = \frac{K}{L}\frac{1}{A(\theta_i,\theta_{i+1})[F(\theta_{i+1})-F(\theta_i)]}$ and it is clear that everything else equal, the higher the transaction cost K, the higher the loading factor.

Fact 2 *If there exists a group $G_i = [\theta_i; \theta_{i+1})$ such that $P(\theta_i, \theta_{i+1}, K) \leq \overline{P}(\theta_i)$, then, all the agents accept the contract $C_i = (P_i, L)$ and the expected profit of the group G_i is equal to zero.*

Let $K > 0$ and consider the design of the first group $[\theta, 1]$, composed with the riskiest agents. From fact 1, we know that the premium $P(\theta, 1, K)$ charged to each agent of the group $[\theta, 1]$ is an increasing function of θ. Since $\overline{P}(\theta)$ is also an increasing functions of θ, it may be the case that there is no $\theta \in (0, 1)$ such that $\overline{P}(\theta) \geq P(\theta, 1, K)$. This may happen when K is too high, see Fig. 1. Consider now the opposite case in which $K = 0$. From equation (18) for a given group $G = [\theta, 1]$ the premium thus reduces to $A(\theta, 1)L$. Since $A(1, 1) = 1$

and $\overline{P}(1) = L$, it thus follows that $A(1,1)L = \overline{P}(1)$. When agents are sufficiently risk-averse and when $A(\theta,1)$ is a convex function of θ, there will exist $\theta < 1$ such that $A(\theta,1)L = \overline{P}(\theta)$. By continuity, this equality will remain true for K low enough. Assume now that there exists $\theta \in (0,1)$ such that $P(\theta,1,K) = \overline{P}(\theta)$. From fact 2, all the agents assigned to the group $G = [\theta,1]$ will accept the contract, and the expected net profit of that group will be equal to zero by construction, i.e., the budget constraint is satisfied. However, when such a $\theta \in (0,1)$ exists, it may fail to be unique, see Fig. 2. In such a case, it is natural to consider the *smallest value* of $\theta < 1$, call it $\theta^{(1)}_{min}$, such that $P(\theta^{(1)}_{min},1,K) = \overline{P}(\theta^{(1)}_{min})$. By doing so, the public insurer is able to design the largest group, that is, that *minimizes* the average transaction cost per policyholder. Note that we obtain *simultaneously* the design of the group and the premium to be charged to each agent of that group. But this is only the first step. Once $\theta^{(1)}_{min} < 1$ is known, one may find try to design another group and so on and so forth. Let $(\theta^{(n)}_{min},...,\theta^{(1)}_{min})$ be an ordered vector such that $\theta^{(i+1)}_{min} < \theta^{(i)}_{min}$. The following definition provides a formal definition of the algorithm used to obtain a segmentation policy.

Definition 2 *The segmentation policy of the public insurer without cross-subsidization is obtained using the following iterative process. Let $\theta^{(0)}_{min}) = 1$ and assume $(\theta^{(n)}_{min},...,\theta^{(1)}_{min})$ is known. At step $n+1$, if it exists, $\theta^{(n+1)}_{min}$ is obtained by solving*

$$\theta^{(n+1)}_{min} = \inf\{\theta \in (0,\theta^{(n)}_{min}) : \overline{P}(\theta) \geq P(\theta,\theta^{(n)}_{min},K)\} \tag{19}$$

and so on and so forth.

To the best of our knowledge, this kind of iterative process has not been considered in the insurance literature. It is however similar in the spirit to the one proposed in Malueg and Schwartz (1994) in a non-insurance framework to maximize the social welfare. There is however an important difference. In Malueg and Schwartz (1994), there is no cost, so that the profit is always non-negative no matter how the prices and the segments are chosen. In our model, even without transaction cost, there is still a positive cost given by the expected cost of claims. Depending on the premium and the market segments chosen by the insurer, the (expected) net profit could be negative. As in Malueg and Schwartz (1994), in our model, the creation of n market segments is a *Pareto improvement* over the case in which there are only $n-1$ groups. Let

$$K_{pub} = \sup\{K \geq 0 : n^*_{pub}(K) \geq 1\} \tag{20}$$

When $K = K_{pub}$, since the market segmentation is the result of an iterative process (see definition 2), it can only be claimed that $n^*_{pub}(K_{pub}) \geq 1$ and this explains the definition of

K_{pub}. In lemma A 3, we show that $K_{pub} < K_{prv}$, which means that when $K < K_{pub}$, both system of insurance are feasible. However, when $K \in (K_{pub}, K_{prv})$, only a private system of insurance can be implemented.

Lemma 2 *Assume that the public insurer implements the iterative process proposed in definition 2. For any positive $K \leq K_{pub}$, there exists a finite integer $\overline{n}(K) \geq 1$ such that equation (19) is not satisfied, that is, the total number of market segments which is designed is necessarily finite.*

Let $\overline{n}(K) = n^*_{pub}(K) := n^{**} \in \mathbb{N}$ be the *socially optimal* number of market segment, defined as the maximum number of market segments which is possible to design using the iterative process. Using the previous notations[8], the (optimal) groups thus are given by $G_1 = [\theta_1^{**}, \theta_2^{**}), ..., G_{n^{**}} = [\theta^{**}_{n^{**}}, 1]$ and note that, as opposed to the private insurer, each θ_i^{**}, $i = 1, ..., n^{**}$ explicitly *depends* on K. To satisfy the budget constraint, the premium of each group must be strictly positive so that $\overline{P}(\theta_1^{**}) > 0$ and as a result, $\theta_1^{**} > 0$. Thus, as long as $K > 0$, agents of the group $[0, \theta_1^{**}]$ are not insured for any choice of $n \geq 1$ so that full participation can not be achieved.

Cross-subsidization. There are actually several ways to introduce cross-subsidization in our framework and the subject raises the following natural question. Who should be subsidized and by whom ? A simple naive answer could state that agents with the highest risk should be subsidized by the richest ones. However, as we explicitly assume that the wealth of each agent is identical, such a wealth argument can not be used. We have seen that as long as K is positive, when using the iterative process, there always exists a fraction of agents that remain without insurance. Since full participation is socially optimal, it thus makes sense for the public insurer to realize some positive profit with those agents that are proposed a contract, and then to use this profit to create a new, subsidized, market segment. In what follows, we offer an approach which is a mix of the two approaches already seen. On the one hand, the public insurer acts as a *profit maximizer*[9] under the constraint given by equation (16). On the other hand, the entire resulting profit, call it $\Pi^\dagger > 0$, is used to create a new group of agents $G_{sub} := [\theta_{sub}, \theta_1]$, where θ_{sub} is, as in the no cross-subsidization case, the smallest value of $\theta < \theta_1$ such that $\mathbb{E}(R_{sub}) = K - \Pi^\dagger$. Since $\Pi^\dagger > 0$ is entirely

[8]Up to an elementary permutation, for each $n \geq 1$, one can move from the ordered vector $(\theta^{(n)}_{min}, ..., \theta^{(1)}_{min})$ to the ordered vector $(\theta_1^{**}, ..., \theta_n^{**})$.

[9]This approach can not, by assumption, maximize the total surplus. However, as it maximizes the total expected profit, it maximizes the amount used to subsidize another group of agents.

used to subsidy the group G_{sub}, the total expected net profit of the public insurer is equal to zero. Note that $\Pi^{\dagger}$ depends on K and is obviously a non-increasing function of K. This approach can thus be seen as an alternative way, with cross-subsidization, to achieve full participation, the socially optimal result.

Definition 3 *The segmentation policy of the public insurer with cross-subsidization is obtained using the following two steps algorithm.*

1. *Solve the profit maximization given by equation (11) under the constraint given by equation (16) and let* $\Pi^{\dagger} > 0$ *be the resulting total net profit.*

2. *Use* $\Pi^{\dagger} > 0$ *to design (if possible) an additional group of the form* $[\theta_{sub}, \theta_1]$.

Let $(\theta_1^{\dagger}, ..., \theta_{n^{\dagger}-1}^{\dagger}, 1)$ be the optimal segmentation policy that results from step 1 and let $G_{sub} = [\theta_{sub}, \theta_1^{\dagger}]$ the new segment that it is possible to create by using the total profit $\Pi^{\dagger} > 0$. Following what has been done before (see equation (18)), this entails to find θ_{sub}, when it exists, such that

$$P(\theta_{sub}, \theta_1^{\dagger}, K, \Pi^{\dagger}) = A(\theta_{sub}, \theta_1^{\dagger})L + \frac{K - \Pi^{\dagger}}{[F(\theta_1^{\dagger}) - F(\theta_{sub})]} = \overline{P}(\theta_{sub}) \tag{21}$$

where $P(\theta_{sub}...)$ is the premium charged to agents of the group G_{sub}. When $K = \Pi^{\dagger}$, everything is as if there were no fixed cost. As a result, this simply means that each agent of the group G_{sub} pays a premium equal to the actuarial cost, equal to the expected cost of claims of that group. Of course, if $\Pi^{\dagger} > K$, this means that the premium paid will be lower than this actuarial cost. At the extreme, it is even possible that $\Pi^{\dagger}$ is so large (e.g., when K is close to zero) that the premium given by equation (21) is equal to zero, i.e., $\theta_{sub} = 0$ by using only a fraction γ of the profit $\Pi^{\dagger}$. In such a case, the remaining profit is equal to $(1 - \gamma)\Pi^{\dagger}$ and is used to be redistributed among the groups.

To distribute the remaining profit, one possibility is to consider the same group and to define a new premium, equal to $P_i'(t_i) = P_i(1 - t_i)$ where $-t_i$ can be seen as a negative tax (i.e., a subsidy). It suffices now to choose t_i, for $i = 1, ..., n^{\dagger}$ such that

$$\Pi^{\dagger}(P_1'(t_1), ..., P_{n^{\dagger}}'(t_{n^{\dagger}})) = (1 - \gamma)\Pi^{\dagger} \tag{22}$$

There are however more than one way to do this. We now consider a simple algorithm for the public insurer to distribute this remaining profit. It works as follows.

1. Set $t_i = 0$ for $i = 1, ..., n^\dagger - 1$ and choose $t_{n^\dagger} \in (0, 1]$. If there exists $t_{n^\dagger} \in (0, 1)$ such that $\Pi^\dagger(P_1, ..., P_{n^\dagger - 1}, P'_{n^\dagger}(t_{n^\dagger})) = (1 - \gamma)\Pi^\dagger$, the problem is solved and the algorithm stops.

2. If not, i.e., if $t_{n^\dagger} = 1$, choose $t_{n^\dagger - 1} \in (0, 1]$. If there exists $t_{n^\dagger - 1} \in (0, 1)$ such that $\Pi^\dagger(P_1, ..., P'_{n^\dagger - 1}(t_{n^\dagger - 1}), P'_{n^\dagger}(1)) = (1 - \gamma)\Pi^\dagger$, the problem is solved and the algorithm stops.

3. and so on and so forth.

It is easy to see that this algorithm will converge after at most $n^\dagger$ iterations. To see this, assume that $t_i = 1$ so that $P'_i(t_i) = 0$ for each $i = 1, ..., n^\dagger$. As a result, $\Pi^\dagger(0, ..., 0) = 0$ and is obviously lower than $(1 - \gamma)\Pi^\dagger > 0$.

Social welfare versus participation : a potential tradeoff. For a given $\lambda > 0$, let $\Psi^\dagger_{pub}(\lambda)$, $(1 - f^\dagger_{pub}(\lambda))$ and $\Psi^*_{pub}(\lambda)$, $(1 - f^*_{pub}(\lambda))$ be the social welfare and the percentage of participation with and without cross-subsidization respectively. In the cross-subsidization case, since the segmentation policy is chosen to maximize the total expected net profit in the first step (i.e., before subsidization), the total surplus is thus necessarily lower than in the no cross-subsidization case. However, it may be the case that the creation of the new group G_{sub} leads to a participation rate which is higher than in the no cross-subsidization, i.e., $f^\dagger_{pub}(\lambda) < f^*_{pub}(\lambda)$ while $\Psi^\dagger_{pub}(\lambda) < \Psi^*_{pub}(\lambda)$. In such a scenario, this generates a tradeoff between social welfare and participation and it would be necessary to introduce an additional criteria to choose the public system to implement, i.e., with or without cross-subsidization.

5 Example and numerical analysis

To perform the numerical analysis, we now consider a more specific model which also allows us to obtain more precised analytical results regarding the optimal market segmentation of the private insurer. We assume that the utility function is equal to $U(W) = \sqrt{W}$ and that the density f is a uniform one, i.e., $f(\theta) = 1$ for all $\theta \in [0, 1]$. Moreover, we assume that $W = L = 1$ so that an agent who is not insured is at risk to loose her entire unit wealth. It is important to note that the qualitative results found in this section are robust to change in the specification of the utility and/of the density. From equation (4) the maximum price agent $\theta \in (0, 1)$ is ready to pay for the full coverage insurance contract is equal to

$$\overline{P}(\theta) = 1 - (1 - \theta)^2 \tag{23}$$

Regarding the transaction cost function, we shall assume that under both insurance system, the distribution cost K is a fraction of Ψ^*_{prv}, the maximum expected profits of the private insurer when there are no transaction cost, that is, $K = \lambda\Psi^*_{prv}$, where $\lambda \in (0,1)$. It is easy to show that under the specific model, $\Psi^*_{prv} = \frac{1}{6} = \Psi^*_{pub}$ so that the transaction cost function reduces to

$$K_n = n\frac{\lambda}{6} \qquad n \geq 1 \tag{24}$$

In what follows, we shall express the two critical thresholds K_{pub} and K_{prv} using λ_{pub} and λ_{prv} respectively.

5.1 Optimal segmentation policies

As in the previous section, we first consider the case of the private insurer and then the case of the public one.

Private insurer. We have already seen that the optimization problem can be decomposed into a two-stage optimization problem. In appendix B, we show that when the density f is a uniform one, the optimal groups must be adjacent. Let us assume that the number $n \geq 1$ of market segments is given. Using equation (37), the optimization problem is reduced to

$$\max_{\vec{\theta}_{(n+1)} \in C([0,1]^{n+1})} \mathbb{E}(R(\vec{\theta}_{(n+1)})) = (\theta_{n+1} - \theta_1) - \sum_{i=1}^{n}(\theta_{i+1} - \theta_i)(1-\theta_i)^2 - \frac{1}{2}(\theta^2_{n+1} - \theta^2_1) \tag{25}$$

While simple in appearance, solving the first order condition of the above optimization problem is still difficult because the partial derivative with respect to θ_i is a non-linear function (indeed a quadratic function) of θ_{i-1}, θ_i and θ_{i+1}. The next proposition allows to go further than proposition 2 and it also provides a simple condition to check that the second order condition is satisfied.

Proposition 4 *For each $n \geq 1$, $\vec{\theta}^*_{(n+1)} = (\theta^*_2, \theta^*_2, ..., \theta^*_{n+1})$ is such that*

1. $\theta^*_i \in \left[\frac{2}{3}\theta^*_{i+1}; \frac{2\theta^*_{i+1}+1}{3}\right)$ $\quad i = 1, ..., n.$
2. *If $\theta^*_{i+1} - \theta^*_i > \theta^*_i - \theta^*_{i-1}$ for $i = 2, ..., n$, then, $\vec{\theta}^*_{(n+1)}$ is the unique maximum.*

The following table reports numerical computations of the candidate point for $n = 1, ..., 7$. One can readily observe that the properties of the candidate points provided in proposition 4 are indeed satisfied. Let $\mathbb{E}(R(\vec{\theta}^*_{(n+1)}) := \phi^*_n$.

Table 1

Optimal market segmentation as a function of $n = 1, 2, ..., 7$									
	θ_1^*	θ_2^*	θ_3^*	θ_4^*	θ_5^*	θ_6^*	θ_7^*	θ_8^*	ϕ_n^*
$n = 1$	0.5	0.75	-	-	-	-	-	-	0.03125
$n = 2$	0.3916	0.5875	0.8298	-	-	-	-	-	0.05685
$n = 3$	0.3141	0.4711	0.6515	0.8785	-	-	-	-	0.07595
$n = 4$	0.25934	0.3891	0.5325	0.698	0.9088	-	-	-	0.090
$n = 5$	0.2198	0.3297	0.4486	0.58	0.7327	0.9286			0.1006
$n = 6$	0.190	0.2853	0.387	0.4965	0.6183	0.7595	0.942		0.109
$n = 7$	0.1674	0.251	0.3395	0.434	0.536	0.649	0.781	0.952	0.115

The following fact provides the optimal number of market segments but also the resulting fraction of agents that are uninsured.

Fact 3 *The optimal number of market segments $n^*_{prv}(\lambda)$ and the fraction of agents uninsured denoted $f^*_{prv}(\lambda)$ are given in the following table as a function of λ.*

λ	*1%*	*2%*	*3%*	*6%*	*7%*	*10%*	*11%*	*13%*	*16%*	*18%*	*19%*
$n^*_{prv}(\lambda)$	*16*	*12*	*8*	*5*	*4*	*3*	*3*	*2*	*1*	*1*	*0*
$f^*_{prv}(\lambda)$	*10.3%*	*14%*	*18.9%*	*29.1%*	*35%*	*43.6%*	*43.6%*	*56%*	*75%*	*75%*	*100%*

The optimal number of market segments turns out to be highly sensitive to the magnitude of transaction costs. When λ increases from 1% to 10%, the optimal number of market segments is reduced from 16 to 3. It is only when λ is high enough, say higher than 10% that we encounter the classical case of two/three market segments. It is easy to show that $\lambda_{prv} = 18.75\%$.

Public insurer : no cross-subsidization. It is easy to show from equation (7) that the average probability of the group G_i is equal to $A(\theta_i, \theta_{i+1}) = \frac{\theta_i + \theta_{i+1}}{2}$. For a given $\lambda > 0$, using equation (18) the premium of the group G_i is equal to

$$P_i = \frac{\theta_i + \theta_{i+1}}{2} + \frac{\lambda}{6(\theta_{i+1} - \theta_i)} \tag{26}$$

To design the first group, i.e., a group of the form $[\theta^{(1)}_{min}, 1]$, one must find the smallest value of θ such that $\frac{1+\theta}{2} + \frac{\lambda}{6(1-\theta)} = 1 - (1 - \theta)^2$. When there is no transaction cost, we have already seen that the insurer can proposed a personalized contract $C(\theta)$ to each agent.

However, the public insurer could also make use of the iterative process. Assume that the public insurer does so when $\lambda = 0$. To design this first group, one must solve $\frac{1+\theta}{2} = 1-(1-\theta)^2$, which is equivalent to solve the quadratic equation $\frac{1}{2} - \frac{3}{2}\theta + \theta^2 = 0$. The unique admissible root is equal to $\theta = \frac{1}{2}$ so that the first group is equal to $[50\%, 100\%]$. To now design the second group, one must solve $\frac{\theta+\frac{1}{2}}{2} = 1 - (1-\theta)^2$, which is equivalent to solve the quadratic equation $\frac{1}{4} - \frac{3}{2}\theta + \theta^2 = 0$. The unique admissible root is equal to $\theta = \frac{3}{4} - \frac{\sqrt{5}}{4} \approx 0.191$ so that the second group is equal to $[19.1\%, 50\%]$. The following fact shows that with six groups, the social result is very close to the one without transaction costs.

Fact 4 *Assume that there are no transaction costs, i.e., $\lambda = 0$ and let $n = 6$. Using the iterative process given in equation (19), the six following groups can be designed, $G_1 = [0.125\%, 0.753\%)$ $G_2 = [0.753\%, 2.25\%)$, $G_3 = [2.25\%, 6.67\%)$, $G_4 = [6.67, \%19.1\%)$, $G_5 = [19.1\%, 50\%)$, $G_6 = [50\%, 100\%]$ and the resulting social welfare is approximately equal to $\Psi^*_{pub} = \frac{1}{6}$.*

It is interesting to note that the group $G_6 = [50\%, 100\%]$, composed with the riskiest agents, already contains 50% of the agents and is the largest group. The premium paid by each agent of that group is equal to $\overline{P}(0.5) = 0.75$ and this means that agents of the subgroup $[50\%, 75\%)$ subsidize agents of the subgroup $[75\%, 100\%]$ since they pay a premium which is higher than their expected loss. The next largest group is the group G_5 that contains approximately 30% of the agents and the smallest one is the group $G_1 = [0.125\%, 0.753\%)$, that contains less than 1% of the agents. The six market segments cases yields a social result very close to the benchmark as the social welfare is almost equal to $\Psi^*_{pub} = \frac{1}{6}$ and only 0.12% of the agents remains uninsured.

Consider now what happens with transaction cost. When $\lambda = 1\%$, only the three following groups can be designed, $G_1 = [8\%, 19.9\%)$, $G_2 = [19.9\%, 50.7\%)$, $G_3 = [50.7\%, 100\%)$ so that 8% of the agents remain uninsured. When $\lambda = 10\%$, only the two following groups can be designed, $G_1 = [29.8\%, 60.9\%)$ and $G_2 = [60.9\%, 100\%)$ and approximately 30% of the agents remain uninsured. Even when $\lambda = 10\%$, the largest groups still contains approximately 40% of the agents. It is interesting to note that the lower bound of the group composed with the riskiest agents is not very sensitive to λ. When λ is multiplied by a factor of 10, i.e., from 1% to 10%, this lower bound (equal to 50.7% when $\lambda = 1\%$ and equal to 60.9% when $\lambda = 10\%$) is only multiplied by 1.2. The following fact provides the socially optimal number of market segments but also the fraction of agents uninsured for few values of λ. The critical threshold $\lambda_{pub} \approx 11.2\%$.

Fact 5 *The socially optimal number of market segments* $n^*_{pub}(\lambda)$ *and the fraction of agents uninsured under the public policy* $f^*_{pub}(\lambda)$ *are given in the following table as a function of* λ.

λ	1%	2%	3%	6%	7%	10%	11%	11.2%
$n^*_{pub}(\lambda)$	3	3	3	2	2	2	2	0
$f^*_{pub}(\lambda)$	8%	9.5%	11.3%	24.5%	25.6%	29.8%	32.2%	100%

The above table shows that, as opposed to the private insurer, the (socially) optimal number of market segments is not very sensitive to transaction cost since this number always remains between 2 and 3. However, as the above table shows, the fraction of agents uninsured is indeed sensitive to the magnitude of the set-up cost. Note that for each value of λ and for each group G_i, $\overline{P}(\theta_i^{**}) < \theta_{i+1}^{**}L$.

5.2 Comparative analysis

When λ is low, we (almost) fall under proposition 1 and the two insurance systems are (almost) equivalent. Cross-subsidization is not an issue. For cross-subsidization to be effective, a non negligible fraction of agents should remain without insurance and this is the case when λ is (relatively) high. In what follows, $\Psi^*_{prv}(\lambda)$, $\Psi^*_{pub}(\lambda)$ and $\Psi^\dagger_{pub}(\lambda)$ will denote the social welfare (as a function of $\lambda > 0$) of the private insurer, public insurer without and with cross-subsidization respectively.

Private versus public insurance system without cross-subsidization.

The numerical results based on this simple model suggest that when λ increases, $f^*_{prv}(\lambda) - f^*_{pub}(\lambda)$ and $\Psi^*_{pub}(\lambda) - \Psi^*_{prv}(\lambda)$ tend to increase. When λ is very small, the two system are (almost) equivalent in terms of exclusion and social welfare as we fall under proposition 1.

Fact 6 *The desirability of the public insurance system increases with* λ.

To illustrate this fact, since $\lambda_{pub} = 11.2\%$, we have chosen to consider $\lambda = 10\%$. As a result, it is always optimal to design two or three market segments, depending on the insurance system under consideration.

The *public insurer* is able to design the two following groups

$$\text{Groups}: \quad G_1 = [29.8\%, 61\%), \; G_2 = [61\%, 100\%) \tag{27}$$

$$\text{Premiums}: \quad P_1 = 0.5072 \qquad P_2 = 0.8479 \tag{28}$$

The *private insurer* is able to design the three following groups

$$\begin{aligned} \text{Groups}: \quad G_1 &= [31.4\%; 47.1\%),\ G_2 = [47.1\%; 65.1\%),\ G_3 = [65.1\%, 87.8\%) \quad (29) \\ \text{Premiums}: \quad P_1 &= 0.5295, \qquad P_2 = 0.72, \qquad P_3 = 0.8827 \quad (30) \end{aligned}$$

It is interesting to note that most of the agents, but not all, pay a lower price under the public insurance system. However, policy holders such that $\theta \in [61\%, 65\%)$ are indeed better off under a private system since the premium they have to pay is equal to 0.72, instead of 0.847 under a public system. From the knowledge of the groups, one immediately obtains that $f^*_{pub}(10\%) = 29.8\%$ and that $f^*_{prv}(10\%) = 43.6\%$. Numerical computations lead to $\Psi^*_{pub}(10\%) = 0.0978$ and $\Psi^*_{prv}(10\%) = 0.07$. If we consider the private insurance system as the current one, moving toward a public system would increase the social welfare by 40% and would decrease the fraction of non insured agents by 30%. Since this numbers, i.e., 40% and 30% are high, this clearly shows that in such a case, the insurance system should be public.

Public insurance system : cross-subsidization versus no cross subsidization. As already seen, to find the optimal market segmentation, we thus solve the profit maximization problem given by equation (25) under the constraint that $\theta_{n+1} = 1$ for each $n \geq 1$. The following table reports the optimal market segmentation for $n = 1, 2, 3$. More is said on the solution on the optimization problem in appendix B. We shall here once again assume that $\lambda = 10\%$.

Table 2

Optimal market segmentation, $n = 1, 2, 3, 4$						
	$\theta_1^\dagger$	$\theta_2^\dagger$	$\theta_3^\dagger$	$\theta_4^\dagger$	$\theta_5^\dagger$	Gross profit
$n = 1$	0.667	1	-	-	-	0.0179
$n = 2$	0.458	0.687	1	-	-	0.04894
$n = 3$	0.345	0.518	0.722	1	-	0.07142
$n = 4$	0.276	0.4141	0.569	0.751	1	0.08728

When $\lambda = 10\%$, the optimal number of market segments is equal to three. From the above table, one can easily derive the groups and the premiums given below.

$$\begin{aligned} \text{Groups}: \quad G_1 &= [34.5\%; 51.8\%),\ [51.8\%; 72.2\%),\ G_3 = [72.2\%, 100\%] \quad (31) \\ \text{Premiums}: \quad P_1 &= 0.571, \qquad P_2 = 0.767, \qquad P_3 = 0.9227 \quad (32) \end{aligned}$$

Compared with a pure profit maximizer insurer, the fraction of agents excluded from the market is much lower, 34.5% instead of 43.6% since the riskiest agents are now insured. The total net profit is equal to $\Pi^* = 0.07142 - \frac{0.1\times 3}{6} = 0.02142$. Using equation (21) with $\theta_1^{**} = 0.345$ in this specific model, we thus have to find the smallest value of θ_{sub} such that

$$P(\theta_{sub}, 0.345) = \frac{\theta_{sub} + 0.345}{2} + \frac{\frac{0.1}{6} - 0.02142}{(0.345 - \theta_{sub})} = 1 - (1 - \theta_{sub})^2 \tag{33}$$

Solving equation (33) leads to $\theta_{sub} = 0.1095$. When the public insurer uses the profit $\Pi^* = 0.02142$ generated to subsidize the creation of a subsidized group G_{sub}, it actually becomes possible to design the group

$$\text{Group}: \quad G_{sub} = [10.95\%; 34.5\%) \tag{34}$$

$$\text{Premium}: \quad P_{sub} = 0.20701 \tag{35}$$

We summarize the results, that is, the fraction of agents excluded and the social welfare.

- Without cross-subsidization
  - Social welfare : $\Psi^*_{pub}(10\%) = 0.0978$
  - Number of market segments : 2
  - Fraction of agents excluded : $f^*_{pub}(10\%) = 29.8\%$.
- With cross-subsidization
  - Social welfare : $\Psi^\dagger_{pub}(10\%) = 0.0927$
  - Number of market segments : 4
  - Fraction of agents excluded : $f^\dagger_{pub}(10\%) = 10.95\%$

These finding are summarized in the following fact.

Fact 7 *When $\lambda = 10\%$, the public insurer faces a tradeoff between the social welfare and participation. Without cross-subsidization, the social welfare is higher than the one with cross-subsidization but fraction of agents excluded is also higher.*

As usual in Economics, to decide whether more participation is better in some sense than higher total surplus, one would have to consider a social welfare function that describes the preferences of the public insurer, i.e., the marginal rate of substitution between total surplus and fraction of agents excluded from the market.

5.3 Robustness check

We performed some robustness check for a model in which

- the density f is a two parameters beta density, i.e., $f(\theta) \propto \theta^{\alpha-1}(1-\theta)^{\beta-1}$, where $\alpha > 0$ and $\beta > 0$ are the two parameters and as usual the symbol $\propto$ means "proportional to".
- the utility function is $U(W) = W^p$, $p \in (0, 1)$.
- a loss $L = \zeta W$, with $\zeta \in (0, 1]$ which is a fraction of the initial wealth.

For full coverage insurance contract, the willingness to pay given by equation (4) becomes, after simple manipulations, equal to

$$\overline{P}(\theta, p, \zeta) = W\left(1 - [\theta(1-\zeta)^p + (1-\theta)]^{\frac{1}{p}}\right) \tag{36}$$

The willingness to pay $\overline{P}(\theta, p, \zeta)$ of a given agent θ is a decreasing function of p and ζ and is equal to equation (23) when $p = 0.5$ and $\zeta = 1$. Everything else equal, the lower is p, the lower is the number of market segments. As long as the premium $\overline{P}(\theta, \overline{\theta})$ a convex function, the results differ quantitatively but not qualitatively. For instance, when the beta density is linear or when it is a monotonic function of θ, the premium turns out to be a convex function of θ and the results are qualitatively similar to those found within the simplest model.

6 Conclusion

We developed in this paper a simple framework of complete information in which an insurer, public or private, decides to segment the market not because of the lack of information about the types, but because of the existence of transaction costs that increase with the number of market segments. For a public insurer, we have shown a fairly general result that states that all the agents should be insured. However, as this general result does not provide any clue to implement this outcome, we considered a simple algorithm to segment the set of policyholders in which there is no-cross-subsidization. We also considered an alternative approach, with cross-subsidization but assuming profit maximization, and it has been shown that the two approaches, with and without cross-subsidization, may generate a tradeoff between total surplus and participation. While we argue that, with the development of technology such as computer mediated transactions (see [Varian, 2010]), it really makes sense today to analyze complete information models, it would also be interesting, from a theoretical point of view, to endogenize the market segmentation when information is incomplete.

Appendix A : Proofs

Proof of Lemma 1. By definition, $\overline{P}(\theta)$ is such that $U(W-\overline{P}(\theta)) = \theta U(W-L) + (1-\theta)U(W)$ so that $\overline{P}(\theta) = W - U^{-1}(\theta U(W-L) + (1-\theta)U(W))$. When θ is equal to 0 or 1, we immediately obtain that $\overline{P}(0) = 0$ and $\overline{P}(1) = L$. Since U is assumed to be strictly concave, $U(\mathbb{E}(X) > \mathbb{E}U(X)$, it is not difficult to show that $\overline{P}(\theta) > \theta L$ for all $\theta \in (0,1)$. Let $C = U(W)-U(W-L) > 0$ and let $G(\theta) = U^{-1}(-\theta C + U(W))$. It is easy to see that $\overline{P}(\theta)$ can be written as $\overline{P}(\theta) = W - G(\theta)$. By direct differentiation, $G'(\theta) = -C(U^{-1}(-\theta C + U(W)))'$ and $G''(\theta) = C^2(U^{-1}(-\theta C + U(W)))''$. Since U is increasing and concave function, U^{-1} is an increasing and convex function[10], it thus follows that $\overline{P}(\theta)$ is an increasing and concave function $\theta \in (0,1)\square$

Proof of proposition 2. Recall from equation (11) that the expected total gross profit is equal to

$$\mathbb{E}R(\vec{\theta}_{(n+1)}) = \sum_{i=1}^{n} \overline{P}(\theta_i)[F(\theta_{i+1}) - F(\theta_i)] - L\int_{\theta_1}^{\theta_{n+1}} \theta f(\theta)d\theta \tag{37}$$

and note that since the total value of the transaction costs depends only on $n \in \mathbb{N}^*$, the MIP can be treated as a two-stage optimization problem. For a given number $n \geq 1$ of market segment(s), the insurer chooses in a first step the optimal market segmentation $\vec{\theta}^*_{(n+1)}$ to maximize the expected total gross profit given by equation (37), and chooses in a second step $n \geq 1$ to maximize $\mathbb{E}\Pi(\vec{\theta}^*_{(n+1)})$. Let $n \in \mathbb{N}^*$ and recall that $C([0,1]^{n+1})$ is a convex and compact cone of $[0,1]^{n+1}$. Since $\mathbb{E}R(\vec{\theta}_{(n+1)})$ is a continuous function of $\vec{\theta}_{(n+1)} \in C([0,1]^{n+1})$, by Wierstrass theorem (see e.g., Bertsekas 1999, proposition A 8 p 654), for each $n \in \mathbb{N}^*$, there exists $\vec{\theta}^*_{(n+1)} = \arg\max_{\vec{\theta}_{(n+1)} \in C([0,1]^{n+1})} \mathbb{E}R(\vec{\theta}_{(n+1)})$. Let $\vec{\theta}^*_{(n+1)} = (\theta^*_1, ..., \theta^*_{n+1}) \in C([0;1]^{n+1})$ be a candidate point to maximize equation (37). Note that all the functions are regular enough by assumption so that it makes sense to compute the gradient and note that $\nabla\mathbb{E}R(\vec{\theta}_{(n+1)}) = \nabla\mathbb{E}\Pi(\vec{\theta}_{(n+1)})$.

Proof of part 1. By direct differentiation, the gradient of equation (37) is given by the

[10] See e.g., the proposition 2 in M Mrešević, (2008), "Convexity of the inverse function", *The teaching of Mathematics*, Vol XI, pp 21-24.

following set of $n+1$ non-linear equations with $n+1$ unknowns.

$$\frac{\partial \mathbb{E}R(\vec{\theta}_{n+1})}{\partial \theta_1} = \overline{P}'(\theta_1)F(\theta_2)) - \overline{P}'(\theta_1)F(\theta_1) - \overline{P}(\theta_1)f(\theta_1) + L\theta_1 f(\theta_1) \quad (38)$$

$$\frac{\partial \mathbb{E}R(\vec{\theta}_{n+1})}{\partial \theta_i} = \overline{P}(\theta_{i-1})f(\theta_i) + \overline{P}'(\theta_i)F(\theta_{i+1}) - \overline{P}'(\theta_i)F(\theta_i) - \overline{P}(\theta_i)f(\theta_i) \; i = 2, ..., n \quad (39)$$

$$\frac{\partial \mathbb{E}R(\vec{\theta}_{n+1})}{\partial \theta_{n+1}} = \overline{P}(\theta_n)f(\theta_{n+1}) - L\theta_{n+1}f(\theta_{n+1}) \quad (40)$$

The first claim, called claim 0 is obvious and we omit the proof.

Claim 0: For each $n \geq 1$, the optimal market segmentation $\vec{\theta}^*_{(n+1)}$ is such that $\theta^*_{i+1} > \theta^*_i$ for all $i = 1, ..., n$.

Claim 1: $\theta^*_1 > 0$. To prove it, assume the contrary is true, i.e., $\theta^*_1 = 0$. Using equation (38), since $\vec{\theta}^*_{(n+1)}$ is a maximum, this means that $\overline{P}'(0)F(\theta^*_2)) - \overline{P}'(0)F(0) - \overline{P}(0)f(0) \leq 0$. Since $F(0) = 0$ and $\overline{P}(0) = 0$, it thus follows that equation (38) is reduced to $\overline{P}'(0)F(\theta^*_2)) \leq 0$. Since $\overline{P}'(0) > 0$ and $F(\theta^*_2) \geq 0$, the unique solution is $\theta^*_2 = 0$, and this contradicts claim 0 $\square$

Claim 2: $\theta^*_n < 1$. To prove this, note since $\theta^*_{n+1} \leq 1$, from claim 0, $\theta^*_n < \theta^*_{n+1} \leq 1$ $\square$

Let us write the candidate point $\vec{\theta}^*_{(n+1)}$ as $\vec{\theta}^*_{(n+1)} = (\vec{\theta}^*_{(n)}; \theta^*_{n+1})$, where $\vec{\theta}^*_{(n)} = (\theta^*_1, ..., \theta^*_n)$. We have shown that $\vec{\theta}^*_{(n)}$ is such that $\theta^*_1 < \theta^*_2 < ... < \theta^*_n < 1$, i.e., it is an interior point of $C([0;1]^n$. It thus follows that $\nabla \mathbb{E}R(\vec{\theta}^*_{(n)}) = \vec{0}$, where $\vec{0}$ is the n-dimensional null vector. To complete the proof, it remains to show that $\frac{\partial \mathbb{E}R(\vec{\theta}^*_{n+1})}{\partial \theta_{n+1}} = 0$.

Claim 3: $\theta^*_{n+1} < 1$ and $\overline{P}(\theta^*_n) = L\theta^*_{n+1}$.

Note that from equation (40), $\theta^*_{n+1} = 1$ can not *a priori* be excluded so that one must explicitly consider the case in which the density is such that $f(1) = 0$ and $f(1) > 0$.

- $f(1) > 0$. To prove the claim, assume the contrary, i.e., that $\theta^*_{n+1} = 1$. Using equation (40), since $\vec{\theta}^*_{(n+1)}$ is a maximum, this means that $f(1)(\overline{P}(\theta^*_n) - L) \geq 0$. Recalling that $\overline{P}(\theta) \leq L$ for all $\theta \in [0,1]$, the unique possibility to satisfy $f(1)(\overline{P}(\theta^*_n) - L) \geq 0$ is when $\overline{P}(\theta^*_n) - L = 0$ and it is the case when $\theta^*_n = 1$. But this contradicts claim 2. It must thus be the case that $\theta^*_{n+1} < 1$ and $\frac{\partial \mathbb{E}R(\vec{\theta}^*_{n+1})}{\partial \theta_{n+1}} = 0$. From equation (40), this implies that $f(\theta^*_{n+1})(\overline{P}(\theta^*_n) - L\theta^*_{n+1}) = 0$, so that $\overline{P}(\theta^*_n) = L\theta^*_{n+1}$.

- $f(1) = 0$. If $\theta^*_{n+1} = 1$, then, equation (40) is always to zero regardless of $\overline{P}(\theta^*_n) - L \leq 0$. Assume that $\overline{P}(\theta^*_n) - L = 0$. Then, $\theta^*_n = 1$ and this contradicts claim 2. Assume now that $\overline{P}(\theta^*_n) - L < 0$. There thus exists $\widetilde{\theta} < 1$ such that $\overline{P}(\theta^*_n) = \widetilde{\theta}L$. But then, for all

$\theta \in (\widetilde{\theta}, 1]$, the premium is lower than the expected loss. As a result, by offering the contract to the subset $[\theta_n^*, \widetilde{\theta}]$, the expected gross profit increases and this contradicts the fact that $\theta_{n+1}^* = 1$ is optimal. It must thus be the case that $\theta_{n+1}^* < 1$ so that $\frac{\partial \mathbb{E}R(\vec{\theta}_{n+1}^*)}{\partial \theta_{n+1}} = 0$ and $\overline{P}(\theta_n^*) = L\theta_{n+1}^*$ □

Remark A 1 *From claim 3, we know that for the group G_n, $\overline{P}(\theta_n^*) = L\theta_{n+1}^*$, that is $\theta_n^* = \overline{P}^{-1}(L\theta_{n+1}^*)$. If $\overline{P}(\theta_i^*) = L\theta_{i+1}^*$ for all $i = 1, ..., n-1$, then, it is also true that $\theta_i^* = \overline{P}^{-1}(L\theta_{i+1}^*)$ so that the maximization problem reduces to the choice of θ_{n+1}^* only since $\theta_{n-1}^* = \overline{P}^{-1}(\overline{P}^{-1}(L\theta_{n+1}^*))$, $\theta_{n-2}^* = \overline{P}^{-1}((\overline{P}^{-1}(\overline{P}^{-1}(L\theta_{n+1}^*)))$ and so on and so forth until θ_1^*. Such a very particular solution should not be true in general. We thus conjecture that for each $n \geq 3$, there exists $i \in \{1, ..., n-1\}$ such that $\overline{P}(\theta_i^*) > L\theta_{i+1}^*$. Consider for instance the case of $n-1$. From the analysis of the gradient (i.e., equation (39) is equal to zero and (40) is equal to zero), it is not difficult to show that $\overline{P}(\theta_{n-1}^*) > L\theta_n^*$ is equivalent to $\frac{\overline{P}'(\theta_n^*)}{Lf(\theta_n^*)}\left[F(\theta_{n+1}^*) - F(\theta_n^*)\right] < \theta_{n+1}^* - \theta_n^*$. By noting that $F(\theta_{n+1}^*) = F(\theta_n^* + (\theta_{n+1}^* - \theta_n^*))$ and by expanding F at order 2, $\frac{\overline{P}'(\theta_n^*)}{Lf(\theta_n^*)}\left[F(\theta_{n+1}^*) - F(\theta_n^*)\right] < \theta_{n+1}^* - \theta_n^*$ is equivalent to $\frac{\overline{P}'(\theta_n^*)}{Lf(\theta_n^*)}\left[f(\theta_n^*) + \frac{1}{2}(\theta_{n+1}^* - \theta_n^*)f'(\theta_c)\right] < 1$ with $\theta_c \in (\theta_n^*, \theta_{n+1}^*)$. No clear conclusion can be drawn without further assumptions. If we further assume that f is uniform, $\overline{P}(\theta_{n-1}^*) > L\theta_n^*$ is equivalent to $\frac{\overline{P}'(\theta_n^*)}{L} < 1$ (since $f = 1$ and $f' = 0$) and this has to be true for n high enough.*

Before we prove part 2, we prove two useful lemma. Let $\mathbb{E}R(\vec{\theta}_{(n+1)}^*) := \phi_n^*$ and let ϕ^* be the maximum total expected gross profit of the private insurer when there are no transaction costs. This quantity has been denoted earlier by Ψ_{prv}^*

Lemma A 1 *There exists $n^* \in \mathbb{N}$ that maximizes $\phi_n^* - nK$*

Proof. Note that $(\phi_n^*)_{n\in\mathbb{N}}$, with $\phi_0^* = 0$, defines a numerical sequence which is strictly increasing with $n \geq 1$ and such that $\phi_n^* < \phi^*$. Let $K_n = nK$. Since $(K_n)_{n\in\mathbb{N}}$, with $K(0) = 0$ defines a sequence which is an increasing function of $n \in \mathbb{N}$ such that $\lim_{n\to\infty} K_n \to \infty$, there exists $n^* \in \mathbb{N}$ that maximizes $\phi_n^* - K_n$. In case of non-uniqueness, as a tie-breaking assumption, we assume that the insurer chooses the smallest value of n^*.

Before we prove that when $K \to 0$, $n^*(K) \to \infty$, we show the convergence of ϕ_n^* to ϕ^* when $K = 0$.

Lemma A 2 $\lim\limits_{n\to\infty} \phi_n^* \to \phi^*$.

Proof. Consider an uniform market segmentation $\vec{\theta}^u$ given by $\theta_i^u = \frac{i}{n+2}$ for $i = 1, 2, ..., n+1$. Let $\phi_n^u := \mathbb{E}(R_n(\vec{\theta}^u)) = W(1 - F(\frac{1}{n+2})) - \sum_{i=1}^{n+1} \left(F(\frac{i+1}{n+2}) - F(\frac{i}{n+2})\right) U^{-1}(\overline{v}(\frac{i}{n+1})) - L \int_{\frac{1}{n+2}}^1 \theta f(\theta) d\theta$. Since F is $\mathcal{C}^2$ on $(0,1)$, it is thus continuous so that $\lim_{n\to\infty} F(\frac{1}{n+2}) \to F(0) = 0$. Since U^{-1} is monotonic and bounded and since F is increasing and continuous, when $n \to \infty$, the term $\sum_{i=1}^{n+1} \left(F(\frac{i+1}{n+2}) - F(\frac{i}{n+1})\right) U^{-1}(\overline{v}(\frac{i}{n+1}))$ converges to $\int_0^1 U^{-1}(v(\theta)) dF(\theta) = \int_0^1 U^{-1}(v(\theta)) f(\theta) d\theta$ (e.g., Rudin (1976) p 126, theorem 6.9), where the equality comes from the fact that F is $\mathcal{C}^1$ (indeed $\mathcal{C}^2$) on $(0,1)$ so that $dF(\theta) = f(\theta)d\theta$. Since it is obvious that when $n \to \infty$, $\int_{\frac{1}{n+2}}^1 \theta f(\theta) d\theta$ converges to $\int_0^1 \theta f(\theta) d\theta$, it thus follows that $\lim_{n\to\infty} \phi_n^u \to \phi^*$. To conclude, it suffices to note that the uniform market segmentation needs not be optimal, so that $\phi_n^* \geq \phi_n^u$, for all $n \in \mathbb{N}$. As a result, $\lim_{n\to\infty} \phi_n^* \to \phi^*$ $\square$

Proof of part 2. We now prove by contradiction that when $K \to 0$, $n^*(K) \to \infty$. Assume that for all $K > 0$, there exists $\overline{n}$ such that $n^*(K) \leq \overline{n}$. Without loss of generality, assume that $n^*(K) = \overline{n}$. This thus means that $\phi_{\overline{n}}^* - \overline{n}K \geq \phi_{\overline{n}+1}^* - (\overline{n}+1)K$ and note that $\phi_{\overline{n}+1}^* - (\overline{n}+1)K \geq \phi_{\overline{n}}^* - \overline{n}K$ is equivalent to $\phi_{\overline{n}+1}^* - \phi_{\overline{n}}^* \geq K$. Since $\phi_{\overline{n}+1}^* - \phi_{\overline{n}}^* > 0$, there exists K small enough such that $\phi_{\overline{n}+1}^* - (\overline{n}+1)K > \phi_{\overline{n}}^* - \overline{n}K$, that is, $n^*(K) \geq \overline{n}+1$ and this contradicts the premises that there exists $\overline{n}$ such that $n^*(K) \leq \overline{n}$ for all $K > 0$ $\square$

Proof of proposition 3. Using equation (6) when $P_i = A(\theta_i, \theta_{i+1})L_i \leq \overline{P}(\theta_i)$, it is not difficult to show that the total surplus $TS(\vec{\theta}_{(n+1)})$ is equal to

$$TS(\vec{\theta}_{(n+1)}) = W(F(\theta_{n+1}) - F(\theta_1)) - \sum_{i=1}^{n} [F(\theta_{i+1}) - F(\theta_i)] A(\theta_i, \theta_{i+1}) L_i - \int_{\theta_1}^{\theta_{n+1}} U^{-1}(v(\theta)) f(\theta) d\theta \tag{41}$$

Assume that the total expected net profit given by equation (13) is equal to zero. It is not difficult to show that $\sum_{i=1}^{n} (L_i - L) A(\theta_i, \theta_{i+1}) [F(\theta_{i+1}) - F(\theta_i)] - nK = 0$ is equivalent to

$$\sum_{i=1}^{n} [F(\theta_{i+1}) - F(\theta_i)] A(\theta_i, \theta_{i+1}) L_i = L \int_{\theta_1}^{\theta_{n+1}} \theta f(\theta) d\theta + nK \tag{42}$$

When one inserts the rhs of equation (42) into equation (41), the total surplus is equal to

$$TS(\vec{\theta}_{(n+1)}) = W(F(\theta_{n+1}) - F(\theta_1)) - \int_{\theta_1}^{\theta_{n+1}} [\theta L + U^{-1}(v(\theta))] f(\theta) d\theta - nK \tag{43}$$

and reduces to a function of θ_1, θ_{n+1}, and $n \geq 1$ only. To simplify the notation, let $TS(\vec{\theta}_{(n+1)}) := \Psi(\theta_1, \theta_{n+1})$. Note that from the above equation, it is easy to see that $\frac{\partial \Psi(\theta_1, \theta_{n+1})}{\partial \theta_1}$ depends only on θ_1 while $\frac{\partial \Psi(\theta_1, \theta_{n+1})}{\partial \theta_{n+1}}$ depends only on θ_{n+1}. As a result, one can examine the optimality condition separately. Let us assume that $n \geq 1$ is given.

Consider first the maximization of $\Psi(\theta_1, \theta_{n+1})$ with respect to θ_1. It is not difficult to see that the first order condition with respect to θ_1 is given by

$$\frac{\partial \Psi(\theta_1^*, \theta_{n+1})}{\partial \theta_1} = 0 \Longleftrightarrow \theta_1^* L + U^{-1}(\overline{v}(\theta_1^*)) - W = 0 \qquad \forall \theta_{n+1} > \theta_1 \tag{44}$$

Since $U^{-1}(\overline{v}(0)) = W$ and $U^{-1}(\overline{v}(1)) = W - L$, it is easy to see that $\theta_1^* = 0$ and $\theta_1^* = 1$ are solutions of equation (44) while $\theta_1^* = 1$ is not a maximum. By concavity of U, for all $\theta \in (0,1)$, $U(W - \theta L) > v(\theta)$, and this is equivalent to $W - \theta L > U^{-1}(v(\theta))$, i.e., $\theta L + U^{-1}(v(\theta)) - W < 0$. As a result, $\frac{\partial \Psi(\theta_1, \theta_{n+1})}{\partial \theta_1} < 0$ for all $\theta_1 \in (0,1)$ and all $n \geq 1$, so that $\theta_1^* = 0$ is a maximum. In the same way, it is now easy to show that the function $\Psi(\theta_1, \theta_{n+1})$ is maximized when $\theta_{n+1} = 1$, i.e., $\theta_{n+1}^* = 1$ □

Proof of lemma 2. For this proof, we change the notations since we assume that the first group is $[\theta_{min}^{(1)}, 1]$, the second group is $[\theta_{min}^{(2)}, \theta_{min}^{(1)})$ etc...that is, for each $n \geq 1$, $\theta_{min}^{(n)} > \theta_{min}^{(n+1)}$ where the recursion is given by equation (19). Assume now that at step $n \geq 1$ of the iterative process, i.e., $n \geq 1$ market segments have been designed so that only the group $[0, \theta_{min}^{(n)}]$ have not been proposed any insurance contract. Note that $(\theta_{min}^{(n)})_{n\geq 1}$ defines a decreasing sequence that converges to zero. At step $n+1$, will possible to design $n+1$ market segments if there exists $\theta_{min}^{(n+1)} < \theta_{min}^{(n)}$, possibly unique, defined as given in equation (19), where, from , equation (18), $P(\theta_{min}^{(n+1)}, \theta_n, K) = A(\theta_{min}^{(n+1)}, \theta_n)L + \frac{K}{[F(\theta_{min}^{(n)}) - F(\theta_{min}^{(n+1)})]}$. Note that when it exists, $\theta_{min}^{(n+1)})$ must be strictly positive since otherwise, the budget constraint would be violated. Since $\theta_{min}^{(n+1)}$ is strictly positive, it is clear that $\frac{K}{[F(\theta_{min}^{(n)}) - F(\theta_{min}^{(n+1)})]} > \frac{K}{F(\theta_{min}^{(n)})}$ so that $P(\theta_{min}^{(n+1)}, \theta_{min}^{(n)}, K) > \frac{K}{F(\theta_{min}^{(n)})}$. Since $(\theta_{min}^{(n)})_{n\geq 1}$ is a decreasing sequence that converges to zero, $\left(\frac{K}{F(\theta_{min}^{(n)})}\right)_{n\geq 1}$ defines an (unbounded) increasing sequence. Since $\left(\overline{P}(\theta_{min}^{(n+1)})\right)_{n\geq 1}$ defines a decreasing sequence that converges to zero, there exists a smallest integer $\overline{n}(K) \equiv \overline{n} \geq 1$ such that $\frac{K}{F(\theta_n)} > \overline{P}(\theta_{min}^{(n+1)})$ for each $n \geq \overline{n}$, i.e., the number of market segments which is possible to design is lower than $\overline{n}$ for each $K > 0$ □

Lemma A 3 $K_{pub} < K_{prv}$

Proof. For simplicity but without loss of generality, assume that $n^{**}(K_{pub}) = 1$ and let $[\theta_1^{**}(K_{pub}); 1]$ be this unique group. By assumption, $\mathbb{E}(R(\theta_1^{**}(K_{pub}), 1)) - K_{pub} = 0$. Consider now the private insurer and assume that $n = 1$. Let $[\theta_1^{**}, \dot{\theta}]$ be the group designed, where $\dot{\theta} > \theta_1^{**}$ is the value of θ that maximizes the expected gross profit of the private insurer. We have already shown that $\dot{\theta} < 1$. As a result, $\mathbb{E}(R(\theta_1^{**}, \dot{\theta})) > \mathbb{E}(R(\theta_1^{**}, 1))$ so that

$\mathbb{E}(\Pi(\theta_1^{**}, \dot{\theta})) > 0$. Since $(\theta_1^{**}, \dot{\theta}))$ needs not be the optimal market segment, i.e., (θ_1^*, θ_2^*), it thus follows that $\mathbb{E}(\Pi(\theta_1^*, \theta_2^*)) \geq \mathbb{E}(\Pi(\theta_1^{**}, \dot{\theta})) > 0$, that is $\mathbb{E}(R(\theta_1^*, \theta_2^*)) - K_{pub} > 0$. As a result, since $K_{prv} > K_{pub}$ $\square$

Proof of proposition 4. Let $\phi(\vec{\theta}_{(n+1)}) = \mathbb{E}(R(\vec{\theta}_{(n+1)}))$ and note that the maximization of ϕ with respect to $\vec{\theta}_{(n+1)}$ is equivalent to the minimization of $g = -\phi$. From equation (25), it thus follows that

$$g(\vec{\theta}_{(n+1)}) = -(\theta_{n+1} - \theta_1) + \sum_{i=1}^{n} (\theta_{i+1} - \theta_i)(1 - \theta_i)^2 + \frac{1}{2}(\theta_{n+1}^2 - \theta_1^2) \tag{45}$$

Let g_i' be the first (partial) derivative of the function g with respect to θ_i and $g_{i,j}''$ be the second (partial) derivative of $g(\vec{\theta}_{(n+1)})$ with respect to θ_i and θ_j. It is easy to show that the gradient of $g(\vec{\theta}_{(n+1)})$ is equal to

$$\begin{aligned} g_1' &= (1 - \theta_1)(3\theta_1 - 2\theta_2) & (46) \\ g_i' &= (1 - \theta_{i-1})^2 - (1 - \theta_i)[2\theta_{i+1} + 1 - 3\theta_i] \quad i = 2, ..., n & (47) \\ g_{n+1}' &= -1 + (1 - \theta_n)^2 + \theta_{n+1} & (48) \end{aligned}$$

Solving $\nabla g(\vec{\theta}^*_{(n+1)}) = \vec{0}$ yields the following system of $n + 1$ non-linear equations with $n + 1$ unknowns.

$$\begin{aligned} \theta_1^* &= \frac{2}{3}\theta_2^* > 0 & (49) \\ (1 - \theta_{i-1}^*)^2 &= (1 - \theta_i^*)[2\theta_{i+1}^* + 1 - 3\theta_i^*] > 0 \quad i = 2, ..., n & (50) \\ \theta_{n+1}^* &= 1 - (1 - \theta_n^*)^2 > 0 & (51) \end{aligned}$$

Remark 1 *As expected from the general analysis of the gradient, in this specific model in which $L = 1$, equation (51) is equivalent to $\overline{P}(\theta_n^*) = \theta_{n+1}^*$.*

Consider the upper bound. From equation (49), since $(1 - \theta_i^*) > 0$ and $(1 - \theta_{i-1}^*)^2 > 0$ for each $i = 2, ..., n$, it must be the case that $[2\theta_{i+1}^* + 1 - 3\theta_i^*] > 0$ and this proves the upper bound. It is clear that $\overline{P}(\theta_i^*) \geq \theta_{i+1}^*$ for each $i = 1, ..., n$, i.e., $1 - (1 - \theta_{i-1}^*)^2 \geq \theta_i^*$. Using now equation (50), $1 - (1 - \theta_{i-1}^*)^2 \geq \theta_i^*$ is equivalent to $1 - (1 - \theta_i^*)[2\theta_{i+1}^* + 1 - 3\theta_i^*] \geq \theta_i^*$ so that $1 > 2\theta_{i+1}^* + 1 - 3\theta_i^*$ and this gives the lower bound $\square$

For $(\theta_1^*, ..., \theta_{n+1}^*)$ to be a (local) minimum of the function g, the Hessian matrix, denoted $H_g(\theta_1^*, ..., \theta_{n+1}^*)$, must be positive definite. Note that in our model, as in all partitioning models, the Hessian matrix is *tridiagonal*. In [Anđelić and Da Fonseca, 2011], they review various interesting results to prove that a tridiagonal matrix is positive definite[11] . We shall actually use their theorem 1.2 which is indeed not specific to tridiagonal matrix and is sometimes known as Gershgorin theorem (or a variant of it). It provides a set of conditions under which a symmetric matrix is positive definite.

Theorem ([Axelsson, 1996] corollary 4.10 p. 133 or [Anđelić and Da Fonseca, 2011] theorem 1.2)). *If $A = (a_{i,j}) \in \mathbb{R}^{n,n}$ is a symmetric matrix with strictly positive diagonal entries (that is, $a_{i,i} > 0$ for all $i = 1, ..., n$) and is moreover strictly diagonal dominant (that is $|a_{i,i}| > \sum_{j \neq i} |a_{i,j}|$ for all $i = 1, ..., n$), then A is positive definite. If A is strictly diagonal dominant with strictly negative entries, then, it is negative definite.*

Lemma A 4 *The Hessian matrix $H_g(\theta_1^*, ..., \theta_{n+1}^*)$ has strictly positive diagonal entries.*

Proof. Let $g''_{i,i}$ denotes the second derivative with respect to θ_i. From equation (54), one obtains that $g''_{1,1} = 3 - 6\theta_1 + 2\theta_2$. Evaluated at $(\theta_1^*, ..., \theta_{n+1}^*)$, since $\theta_1^* = \frac{2}{3}\theta_2^*$, $g''_{1,1} = 3 - 2\theta_2^*$ and thus is strictly positive since $\theta_2^* < 1$. From equation (55), evaluated at $(\theta_1^*, ..., \theta_{n+1}^*)$, $g''_{i,i} = 2(2 + \theta_{i+1}^* - 3\theta_i^*)$ for $i = 2, ..., n$. From the positivity of the rhs of equation (50), we know that $(2\theta_{i+1}^* + 1 - 3\theta_i^*) > 0$. Since $2 + \theta_{i+1}^* - 3\theta_i^* > 2\theta_{i+1}^* + 1 - 3\theta_i^*$ when $\theta_{i+1}^* < 1$, it thus follows that $(2 + \theta_{i+1}^* - 3\theta_i^*) > 0$ since $\theta_{i+1}^* < 1$ so that $g''_{i,i} > 0$ for all $i = 2, ..., n$. Since $g''_{n+1,n+1} = 1$, the result follows $\square$

Remark 2 *Since $\theta_i^* > \theta_{i-1}^*$, $(1 - \theta_{i-1}^*) > (1 - \theta_i^*)$ so that $(1 - \theta_{i-1}^*)^2 > (1 - \theta_i^*)(1 - \theta_{i-1}^*)$. From equation (50), for $i = 2, ..., n$, it thus follows that $(1 - \theta_{i-1}^*) < 2\theta_{i+1}^* + 1 - 3\theta_i^*$ which is in turn is equivalent to $\theta_{i+1}^* - \theta_i^* > \frac{1}{2}(\theta_i^* - \theta_{i-1}^*)$.*

Lemma A 5 *Let $n \geq 2$. If $\vec{\theta}_{(n+1)}^*$ is such that $\theta_{i+1}^* - \theta_i^* > \theta_i^* - \theta_{i-1}^*$ for $i = 2, ..., n$, then,*

1. *$H_g(\theta_1^*, ..., \theta_{n+1}^*)$ is strictly diagonal dominant*
2. *$\vec{\theta}_{(n+1)}^*$ is the unique maximum.*

Proof of part 1.. We already know that evaluated at the point $(\theta_1^*, ..., \theta_{n+1}^*)$, $g''_{1,1} = 3 - 2\theta_2^* > 0$. Since $g''_{1,2} = -2 + 2\theta_1 < 0$ (i.e., $|g''_{1,2}| = 2 - 2\theta_1$), the diagonal dominance condition

[11] Some of the results are interesting but we have not found them suitable for our model. If $A = (a_{i,j}) \in \mathbb{R}^{n,n}$ is a tridiagonal matrix, few theorems presented show that the positive definitness of A turns out to be related to $\frac{1}{\cos^2(\frac{\pi}{n+1})}$.

for the first row thus is $g''_{1,1} > |g''_{1,2}|$, which is equivalent to the positivity of $3 - 2\theta_2^* - 2 + 2\theta_1^*$. Since $\theta_1^* = \frac{2}{3}\theta_2^*$, the positivity is always satisfied. From equation (55), one easily obtains that $g''_{i,i-1} = -2(1-\theta_{i-1}) < 0$, $g''_{i,i+1} = -2(1-\theta_i) < 0$. Since $g''_{i,i} = 2(2+\theta_{i+1} - 3\theta_i)$ for $i = 2, ..., n-1$, evaluated at $(\theta_1^*, ..., \theta_{n+1}^*)$, the diagonal condition thus is $2(2+\theta_{i+1}^* - 3\theta_i^*) > 2(1-\theta_i) + 2(1-\theta_{i-1})$ and is satisfied if $\theta_{i+1}^* - \theta_i^* > \theta_i^* - \theta_{i-1}^*$. For $i = n$, one easily obtains that $g''_{n+1,n+1} = 1$ and $g''_{n+1,n} = -2(1-\theta_n^*)$ so that the diagonal condition is satisfied if $\theta_n^* > 1/2$. We already know that... Since for $n = 1$, $\theta_1^* = 1/2$, it thus follows that for $n \geq 2$, $\theta_n^* > 1/2$. From Gershgorin theorem, $H_g(\theta_1^*, ..., \theta_{n+1}^*)$ is positive definite so that $\vec{\theta}^*_{(n+1)}$ is a minimum of g and thus a maximum of ϕ □

Proof of part 2.. It now remains to show that $\vec{\theta}^*_{(n+1)}$ is unique. To do so, assume the contrary. For simplicity, assume that there are two vectors denoted $\vec{\theta}_a^* = (\theta_{a,1}^*, ..., \theta_{a,n+1}^*)$ and $\vec{\theta}_b^* = (\theta_{b,1}^*, ..., \theta_{b,n+1}^*)$ of $C([0,1]^{n+1})$ that minimize the function g, that is, $\vec{\theta}_a^*$ and $\vec{\theta}_b^*$ satisfy the first order condition and are such that $g(\vec{\theta}_a^*) = g(\vec{\theta}_b^*) < g(\vec{\theta}_{(n+1)})$ for all $\vec{\theta}_{(n+1)} \in C([0,1]^{n+1}) \backslash \{\vec{\theta}_a^*; \vec{\theta}_b^*\}$. Let

$$\mathcal{S} = \{\vec{\theta}_{(n+1)} \in C([0,1]^{n+1}) : \vec{\theta}_{(n+1)} = \lambda\vec{\theta}_a^* + (1-\lambda)\vec{\theta}_b^*, \lambda \in (0,1)\} \tag{52}$$

that is, the set of points which are convex combination of $\vec{\theta}_a^*$ and $\vec{\theta}_b^*$. Since $C([0,1]^{(n+1)})$ is a convex cone, it thus follows that $\mathcal{S} \subset C([0,1]^{(n+1)})$. It is not difficult to show that if $\theta_{i+1} - \theta_i > \theta_i - \theta_{i-1}$ for $i = 2, ..., n$ is satisfied for $\vec{\theta}_a^*$ and $\vec{\theta}_b^*$, it also holds for each vector of $\vec{\theta}_{(n+1)} \in \mathcal{S}$. But this means for each $\vec{\theta}_{(n+1)} \in \mathcal{S}$, the Hessian matrix is positive definite, which is equivalent to the strict convexity of the function $g(\vec{\theta}_{(n+1)})$ for $\vec{\theta}_{(n+1)} \in \mathcal{S}$. And this contradicts the possibility of two minima □

Proposition A 1 *For any choice of $n \geq 1$, maximizing equation (25) subject to the condition $\theta_{n+1} = 1$ for each $n \geq 1$ leads to the existence of positive numbers $(\nu_i)_{i=1}^n$ such that $\theta_i^\dagger = 1 - \nu_i \left(\frac{1}{3\nu_1 - 2\nu_2}\right)$ for $i = 1, ..., n$.*

Proof. We use the objective function defined in equation (25) with $\theta_{n+1} = 1$ for each $n \geq 1$.

$$f(\theta_1, ..., \theta_n) = (1-\theta_1) - \sum_{i=1}^{n} (\theta_{i+1} - \theta_i)(1-\theta_i)^2 - \frac{1}{2}(1-\theta_1^2) \tag{53}$$

Let f'_i be the first (partial) derivative of the function f with respect to θ_i. It is not difficult to show that the gradient of $f(\theta_1, ..., \theta_n)$ is equal to

$$f_1' = -(1-\theta_1) + (1-\theta_1)^2 + 2(1-\theta_1)(\theta_2 - \theta_1) \tag{54}$$

$$f_i' = -(1-\theta_{i-1})^2 + (1-\theta_i)^2 + 2(1-\theta_i)[(1-\theta_i) - (1-\theta_{i+1})] \quad i = 2, ..., n-1 \tag{55}$$

$$f_n' = -(1-\theta_{n-1})^2 + 3(1-\theta_n)^2 \tag{56}$$

For a given $n \geq 1$, let the vector $(\theta_1^\dagger, \theta_2^\dagger, ..., \theta_n^\dagger) \in C([0,1]^n)$ be such that $f_i' = 0$ for each $i = 1, ..., n$.

Lemma A 6 *For each $n \geq 1$, there exists strictly positive numbers $\nu_0, \nu_1, ..., \nu_n$ such that $(1 - \theta_{n-k}^\dagger) = \nu_{n-k}(1 - \theta_n^\dagger)$.*

Proof. For $k = 0$, it is obvious that $\nu_n = 1$. For $k = 1$, by using equation (56), it is easy to see that $f_n' = 0$ is equivalent to $(1 - \theta_{n-1}^\dagger) = \sqrt{3}(1 - \theta_n^\dagger)$ and as a result, $\nu_{n-1} = \sqrt{3}$. For $k = 2$ (i.e., to obtain ν_{n-2}), consider equation (55) when $f_{n-1}' = 0$ for $i = n-1$. One obtains that $(1-\theta_{n-2}^\dagger)^2 = 3(1-\theta_{n-1}^\dagger)^2 - 2(1-\theta_{n-1}^\dagger)(1-\theta_n^\dagger)$. Since $(1-\theta_{n-1}^\dagger) = \nu_{n-1}(1-\theta_n^\dagger)$, it thus follows that $(1-\theta_{n-2}^\dagger)^2 = 3(1-\theta_{n-1}^\dagger)^2 - 2(1-\theta_{n-1}^\dagger)(1-\theta_n^\dagger)$. This can be written as $(1-\theta_{n-2}^\dagger)^2 = (3\nu_{n-1}^2 - 2\nu_{n-1})(1-\theta_n^\dagger)^2$, which leads to $\nu_{n-2} = \sqrt{(3\nu_{n-1}^2 - 2\nu_{n-1})}$. Let $(1-\theta_{n-k}^\dagger) = \nu_{n-k}(1-\theta_n^\dagger)$ and consider the case in which $i = n-k$, for $k = 3..., n-2$ in equation (55) when $f_{n-k}' = 0$. By plugging $(1-\theta_{n-k}) = \nu_{n-k}(1-\theta_n)$ into equation (55), after some manipulations, we obtain that $\nu_{n-k-1} = \sqrt{3\nu_{n-k}^2 - 2\nu_{n-k}^2\nu_{n-k+1}^2}$. It remains to consider the case in which $n = k-1$. It suffices to insert $(1-\theta_1^\dagger) = \nu_1(1-\theta_n^\dagger)$ into equation (54) for $f_1' = 0$. Since $\theta_n^\dagger < 1$, after simplifications, we obtain that $\theta_n^\dagger = 1 - \frac{1}{3\nu_1 - 2\nu_2}$. □

We have shown that

$$(1-\theta_i^\dagger) = \nu_i(1-\theta_n^\dagger) \quad for \quad i = 1, ..., n \tag{57}$$

It suffices to insert $\theta_n^\dagger = 1 - \frac{1}{3\nu_1 - 2\nu_2}$ into equation (57) to obtain the following corollary.

Corollary A 1 $\theta_i^\dagger = 1 - \frac{\nu_i}{3\nu_1 - 2\nu_2}$ *for $i = 1, ..., n$*

This concludes the proof □

In the special case in which $n = 3$, this gives $\nu_3 = 1$, $\nu_2 = \sqrt{3}$ and $\nu_1 = \sqrt{9 - 2\sqrt{3}}$. As a result, $\theta_1^\dagger = 1 - \frac{\sqrt{9-2\sqrt{3}}}{3\sqrt{9-2\sqrt{3}} - 2\sqrt{3}} \approx 0.3454$, and $\theta_2^\dagger$ and $\theta_3^\dagger$ can be computed in the same way. While tedious, it is not difficult to repeat this analysis for different values of $n > 3$.

Appendix B

In this appendix, we consider the possibility to design non-adjacent groups for a private insurer and the possibility to offer partial insurance for a public insurer.

7 Allowing groups to be non-adjacent

We consider the private insurer. Let $(\underline{\theta}_i, \overline{\theta}_i)$ defines the group G_i for $i = 1, ..., n$ with $\overline{\theta}_i \leq \overline{\theta}_{i+1}$. We now derive few simple results on the optimal market segmentation given by $(\underline{\theta}_i^*, \overline{\theta}_i^*)$ for $i = 1, ..., n$. For some results, the assumption that f is continuous is even not required.

When market segments are assumed to be adjacent, the optimal market segmentation is the solution of an *unconstrained* optimization problem in which the insurer has to choose n variables, i.e., $(\theta_1, \theta_2, ..., \theta_n)$. When segments are allowed to be non-adjacent, the optimal market segmentation is the solution of a *constrained* optimization problem in which the insurer has to choose $2n$ variables, i.e., $(\underline{\theta}_i, \overline{\theta}_i)$ for $i = 1, ..., n$ subject to the constraint $\underline{\theta}_{i+1} - \overline{\theta}_i \geq 0$. The real difficulty is of course to assess whether a given constraint $\underline{\theta}_{i+1} - \overline{\theta}_i \geq 0$ is binding or not.

Properties of the optimal market segmentation

Fact A 1 *Under the optimal market segmentation* $(\underline{\theta}_i^*, \overline{\theta}_i^*)$, $i = 1, ..., n$, *the expected profit of each market segment* i, $\mathbb{E}\pi(\underline{\theta}_i^*, \overline{\theta}_i^*)$ *is positive.*

Proof. Assume the contrary is true, that is, there is an index i such that $\mathbb{E}\pi(\underline{\theta}_i^*, \overline{\theta}_i^*) < 0$. In such a case, it suffices not to offer a contract to the market segment i. As a result, the profit of the segment i is now equal to zero (there are now $n-1$ market segments) and the total expected profit increases, which contradicts the optimality of the market segmentation □

Since the expected profit of each group is positive under the optimal market segmentation, the premium of each group has to be positive.

Corollary A 2 *For* $i = 1, ..., n$, *the premium of the group* G_i, $\overline{P}(\underline{\theta}_i^*)$, *is positive so that* $\underline{\theta}_i^*$ *is positive.*

Fact A 2 $\overline{P}(\underline{\theta}_n^*) = \overline{\theta}_n^* L$ *and* $\overline{\theta}_n^* < 1$

See the proof of claim 3 of proposition 2.

Fact A 3 *For $i = 1, ..., n$, $\overline{P}(\underline{\theta}_i^*) \geq \overline{\theta}_i^* L$.*

Proof. Assume the contrary is true for a given group i, that is, $\overline{P}(\underline{\theta}_i^*) < \overline{\theta}_i^* L$. By continuity of the function $\overline{P}(\theta)$, there exists $\widetilde{\theta}_i \in (\underline{\theta}_i^*, \overline{\theta}_i^*)$ such that $\overline{P}(\underline{\theta}_i^*) = \widetilde{\theta}_i L$. Since the expected (gross) profit is negative on the subgroup $[\widetilde{\theta}_i, \overline{\theta}_i^*]$, the expected profit increases by not offering a contract to this subgroup and this contradicts the optimality of $\overline{\theta}_i^*$ □

Fact A 4 *Let G_i and G_{i+1} be two groups under the optimal market segmentation.*

1. *If G_i and G_{i+1} are not-adjacent, then, $\overline{P}(\underline{\theta}_i^*) = \overline{\theta}_i^* L$.*
2. *If G_i and G_{i+1} are adjacent, then, $\overline{P}(\underline{\theta}_i^*) \geq \overline{\theta}_i^* L$.*

Proof. Part 1. Let $G_i = [\underline{\theta}_i^*, \theta]$ and assume that everything is (optimally) chosen except θ, the upper bound of that group. The premium paid by the group $G_i = [\underline{\theta}_i^*, \theta]$ is equal to $\overline{P}(\underline{\theta}_i^*)$ and θ has to be chosen. Since the expected profit of the group G_i is an increasing function of θ as long as θ is such that $\overline{P}(\underline{\theta}_i^*) \leq \theta L$, it thus follows that the optimal value of θ, denoted by $\overline{\theta}_i^*$, is such that $\overline{P}(\underline{\theta}_i^*) = \overline{\theta}_i^* L$ since the optimal groups G_i and G_{i+1} are not adjacent by assumption, i.e., $\overline{\theta}_i^* < \underline{\theta}_{i+1}^*$. Part 2 see the proof of fact A 3 . □

From an economic point of view, it thus follows from the above results, that for each $n \geq 1$, the subset of agents $[0, \underline{\theta}_1^*)$ and the subset of agents $[\overline{\theta}_n^*, 1]$ are never proposed an insurance contract so that they remain uninsured. However, if there exists an index $i \in \{1, ..., n-1\}$ such that G_i and G_{i+1} are not adjacent under the optimal market segmentation, then, the subset of agents $(\overline{\theta}_i^*, \overline{\theta}_{i+1}^*)$ also remain uninsured. As said, the difficulty lies on the assessment of whether or not a constraint i is binding or not.

On the possibility of optimal groups that are not adjacent

Consider first the case of a single group $G = [\underline{\theta}, \overline{\theta}]$ and let $(\underline{\theta}^*, \overline{\theta}^*)$ be the optimal market segmentation, i.e., that maximizes $\mathbb{E}R(\underline{\theta}, \overline{\theta})$

Lemma A 7 *Assume that $n = 1$. If f is $\mathcal{C}^1$ on $(0, 1)$, then, $\nabla \mathbb{E}R(\underline{\theta}^*, \overline{\theta}^*) = \vec{0}$ leads to*

$$[\overline{P}'(\underline{\theta}^*) - L] f(\underline{\theta}^*) = -\frac{1}{2} f'(\theta_c)(\overline{\theta}^* - \underline{\theta}^*) \overline{P}'(\underline{\theta}^*) \quad \theta_c \in (\underline{\theta}^*, \overline{\theta}^*) \tag{58}$$

Moreover, if f is strictly increasing (decreasing), then $\underline{\theta}^* > \widehat{\theta}$ ($\underline{\theta}^* < \widehat{\theta}$), where $\widehat{\theta} \in (0, 1)$ is the unique value of θ such that $\overline{P}'(\widehat{\theta}) = L$,

Proof. Note that, up to the notations, the gradient is given by equations (38) and (40) with $n = 1$. We have already seen that $\overline{P}(\underline{\theta}^*) = \overline{\theta}^* L$, with $\overline{\theta}^* < 1$. Plugging $\overline{P}(\underline{\theta}^*) = \overline{\theta}^* L$ into equation (38) yields to $\overline{P}'(\underline{\theta}^*)[F(\overline{\theta}^*) - F(\underline{\theta}^*)] = f(\underline{\theta}^*)[(\overline{\theta}^* - \underline{\theta}^*)L$. Noting that $F(\overline{\theta}^*) = F(\underline{\theta}^* + (\overline{\theta}^* - \underline{\theta}^*))$, since F is assumed to be $\mathcal{C}^2$, $F(\overline{\theta}^*) - F(\underline{\theta}^*) = (\overline{\theta}^* - \underline{\theta}^*)f(\underline{\theta}^*) + \frac{1}{2}(\overline{\theta}^* - \underline{\theta}^*)^2 f'(\theta_c)$, where $\theta_c \in (\underline{\theta}^*, \overline{\theta}^*)$. By inserting $F(\overline{\theta}^*) - F(\underline{\theta}^*)$ in equation (38), this proves, equation (58).

Since $\overline{P}(\theta)$ is an increasing and (strictly) concave function of θ such $\overline{P}(\theta) > \theta L$ for $\theta \in (0, 1)$, with $\overline{P}(0) = 0$ and $\overline{P}(1) = L$, there exists a unique $\widehat{\theta} \in (0, 1)$ such that $\overline{P}'(\widehat{\theta}) = L$, i.e., for all $\theta \in (0, \widehat{\theta})$, $\overline{P}'(\theta) > L$ while for all $\theta \in (\widehat{\theta}, 1)$, $\overline{P}'(\theta) < L$. Assume now that f is strictly increasing, i.e., $f' > 0$ for all $\theta \in (0, 1)$. If $\underline{\theta}^* \leq \widehat{\theta}$, then, the lhs of equation (58) is positive (or equal to zero) while the rhs is negative and is impossible. As a result, it is only when $\underline{\theta}^* > \widehat{\theta}$ that equation (58) can be satisfied. Similar arguments show that $\underline{\theta}^* < \widehat{\theta}$ when f is strictly decreasing $\square$

Consider now the case in which $n = 2$ and assume that the insurer can choose market segments which may not be adjacent intervals. Let $G_1 = [\underline{\theta}_1, \overline{\theta}_1]$ and $G_2 = [\underline{\theta}_2, \overline{\theta}_2)$ where $\overline{\theta}_1 \leq \underline{\theta}_2$ be two such groups. Let $\mathbb{E}\Pi(\underline{\theta}_1, \overline{\theta}_1, \underline{\theta}_2, \overline{\theta}_2) := \mathbb{E}R_1(\underline{\theta}_1, \overline{\theta}_1) + \mathbb{E}R_2(\underline{\theta}_2, \overline{\theta}_2)$ be the expected total gross profit where $\mathbb{E}R_i(...)$, $i = 1, 2$ is given by equation (8). As already said, one can formulate the problem as the following optimization problem.

$$\max_{\underline{\theta}_1, \overline{\theta}_1, \underline{\theta}_2, \overline{\theta}_2} \mathbb{E}R_1(\underline{\theta}_1, \overline{\theta}_1) + \mathbb{E}R_2(\underline{\theta}_2, \overline{\theta}_2) \tag{59}$$

$$s/c \quad \underline{\theta}_2 - \overline{\theta}_1 \geq 0 \tag{60}$$

As it is a constrained optimization problem, it can be formulated using KKT. Let μ be the multiplier associated to the constraint. This is written $\mu(\underline{\theta}_2 - \overline{\theta}_1) = 0$, that is, $\mu = 0$ if the constraint is not binding and $\mu > 0$ if the constraint is binding.

Assume now that under the optimal market segmentation, the two optimal groups are not adjacent, i.e., $\mu = 0$ since the constraint is not binding. This means that the constraint is not effective, and that there exists $0 < \underline{\theta}_1^* < \overline{\theta}_1^* < \underline{\theta}_2^* < \overline{\theta}_2^* < 1$ such that equation (58) holds for $(\underline{\theta}_1^*, \overline{\theta}_1^*)$ and $(\underline{\theta}_2^*, \overline{\theta}_2^*)$ *separately.*

Proposition A 2 *When f is the uniform density, the optimal market segmentation chosen by the profit maximizing insurer must be composed of adjacent market segments.*

Assume that the two optimal groups are not adjacent and that f is the uniform density. In such a case, the rhs of equation (58) is equal to zero since $f'(\theta) = 0$ for all $\theta \in (0, 1)$. It thus follows that for $i = 1, 2$, $[\overline{P}'(\underline{\theta}_i^*) - L]f(\underline{\theta}_i^*) = 0$, so that $\overline{P}'(\underline{\theta}_i^*) = L$ since $f(\theta) = 1$ for all

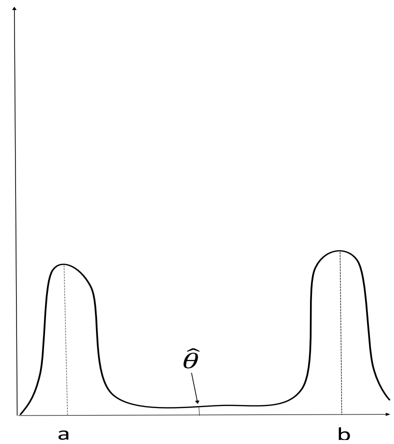

Figure 3: Example of a bimodal distribution

$\theta \in (0,1)$. As a result, $\underline{\theta}_i^* = \widehat{\theta}$ for $i = 1,2$. Since $\widehat{\theta}$ is unique, this thus means that $\underline{\theta}_1^* = \underline{\theta}_2^*$ so that the two groups are overlapping and this contradicts the premises that the groups are non adjacent. Since this is true for an arbitrary number of groups, this concludes the proof of the proposition □

We shall now show that a typical example in which market segments are not adjacent is when the density is bimodal. On Fig. (3), a and b are two local max of the density.

Lemma A 8 *Assume that the density is bimodal as in Fig. (3) and let $n = 2$. Let $G_1 = [\underline{\theta}_1^*, \overline{\theta}_1^*]$ and $G_2 = [\underline{\theta}_2^*, \overline{\theta}_2^*]$ be the two optimal groups and let $\theta_{i,c} \in (\underline{\theta}_i^*, \overline{\theta}_i^*)$, $i = 1,2$. If $\overline{\theta}_1^* < \widehat{\theta}$ and if $\underline{\theta}_2^* > \widehat{\theta}$ and $\overline{\theta}_2^* \leq b$, $(\underline{\theta}_i^*, \overline{\theta}_i^*)$ may solve the first order condition given by equation (58) for each $i = 1,2$.*

Proof. Recall that the first order condition is given by

$$[\overline{P}'(\underline{\theta}^*) - L]f(\underline{\theta}^*) = -\frac{1}{2}f'(\theta_c)(\overline{\theta}^* - \underline{\theta}^*)\overline{P}'(\underline{\theta}^*) \quad \theta_c \in (\underline{\theta}^*, \overline{\theta}^*) \tag{61}$$

Since $\overline{\theta}_1^* < \widehat{\theta}$, it thus follows that $\underline{\theta}_1^* < \widehat{\theta}$ so that $[\overline{P}'(\underline{\theta}_1^*) - L] > 0$. If $\theta_{1,c} \in (a, \widehat{\theta})$, see Fig. (3), $f'(\theta_{1,c})$ is negative so that equation (61) may be satisfied. Since $\underline{\theta}_2^* > \widehat{\theta}$, it thus follows that $[\overline{P}'(\underline{\theta}_2^*) - L] < 0$. Since $\overline{\theta}_2^* \leq b$, see Fig. (3), $f'(\theta_{2,c}) > 0$ so that equation (61) may be satisfied □

A similar result can be obtained when $n = 2$ when the density is U-shaped. From this analysis, it is quite clear that the conditions under which two market segments are non-adjacent are rather particular.

8 On partial coverage public insurance contracts

Partial insurance contracts through a deductible

From [Mossin, 1968], we know that an expected utility maximizer agent θ should optimally choose a partial insurance coverage when the premium is actuarially unfavorable. Formally, if an agent θ is assigned to the group $G_i = [\theta_i, \theta_{i+1})$ and if the premium $P_i > \theta L$, then, there exists $c(\theta) > 0$ such that $P_i = (1 + c(\theta))\theta L$ where $c(\theta)$ can be interpreted as the implied loading factor for agent θ. Following [Razin, 1976], the [Mossin, 1968] result can be formulated by incorporating a deductible $D \leq L$ in the contract, possibly chosen by the policy holder θ. In such a case, the premium can written as $P_i(D) = (1 + c(\theta))\theta(L - D)$ and it is clear that the no deductible case, i.e., $D = 0$, is equivalent to complete coverage. With a deductible, the contract offered to a group G_i is $C_i = (P_i(D), L - D)$ where $L - D$ is the payment of the insurer in case of damage. It is shown in [Razin, 1976] (see also [Briys and Louberge, 1985]) that if the loading factor is positive, i.e., $c(\theta) > 0$, then, an expected utility maximizer would optimally choose a positive deductible, i.e., $D^*(\theta) > 0$. To examine whether or not a public insurer should only offer partial coverage contracts, assume that the public insurer has designed the group $G_i = [\theta_i, \theta_{i+1})$. With transaction costs, to satisfy the budget constraint of that group, the premium P_i must clearly be higher than $\theta_i L$. From equation (18), depending on the inputs of the model (density, utility function...), it may be the case that there is an index i such that $\overline{P}(\theta_i) > \theta_{i+1} L$. In such a case, it might be worthwhile for the public insurer to offer a partial coverage insurance contract to all agents of this group. There are thus two cases

1. $\overline{P}(\theta_i) < \theta_{i+1} L$.

2. $\overline{P}(\theta_i) > \theta_{i+1} L$.

Consider first the case in which $\overline{P}(\theta_i) < \theta_{i+1} L$ and note that there exists $\widehat{\theta}_i \in (\theta_i; \theta_{i+1})$ such that $\overline{P}(\theta_i) = \widehat{\theta}_i L$. In such a case, the contract is actuarially favorable to all the agents of the subgroup $(\widehat{\theta}_i, \theta_{i+1})$ and they find optimal to be fully insured. However, all the agents of the subgroup $[\theta_i, \widehat{\theta}_i)$ find the contract actuarially unfavorable and would have preferred a partial coverage insurance contract. In this case, whether or not a partial insurance coverage will increase the aggregate surplus of the group is unclear. Consider now the case in which $\overline{P}(\theta_i) > \theta_{i+1} L$, that is, all the agents of the group G_i find the contract actuarially unfavorable. In such a case, offering a partial insurance coverage might be welfare increasing for such a group.

Formulating the optimization problem

We here assume that the partial (public) insurance contract is designed through a deductible arrangement $D < L$. Consider a given potential policy holder θ and a given deductible $D < L$. Let $\overline{P}(\theta, D)$ be the maximum price agent θ is ready to pay for such a partial insurance contract, that is, that pays only $L - D$ in case of damage. Recalling that $v(\theta)$ is the expected utility without insurance, $\overline{P}(\theta, D)$ is the solution of the following equation (i.e., the expected utility)

$$\theta U(W - \overline{P}(\theta, D) - D) + (1 - \theta)U(W - \overline{P}(\theta, D)) = v(\theta) \tag{62}$$

so that $\overline{P}(\theta, D)$ can only be defined *implicitly*. It is only in the no-deductible case that there is an explicit expression for $\overline{P}(\theta, D)$. We have seen that when $P_i > \theta_{i+1}L$, all the agents of the group G_i would be prefer a partial coverage contract. Let $P_i(D)$ be the premium offered to the group G_i and assume that all the agents accept the contract. The aggregate surplus thus is equal to

$$CS_i(D) = \int_{\theta_i}^{\theta_{i+1}} \left(\overline{P}(\theta, D) - P_i(D)\right) f(\theta) d\theta \tag{63}$$

For a given group $G_i = [\theta_i, \theta_{i+1})$, the optimization problem of the public insurer can thus be formulated as follows.

$$\max_{D \geq 0} \; CS_i(D) \tag{64}$$

$$s/c$$

$$\overline{P}(\theta, D) \geq P_i(D) \quad \forall \theta \in G_i$$

$$\mathbb{E}(R_i(D) = K$$

Even when one specifies the model, this problem has to be solved numerically.

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