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On Estimating Optimal α-Returns to Scale

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Abstract

From a theoretical point of view, α-returns to scale is a relevant alternative to traditional Data Envelopment Analysis models for estimating production technologies under global returns to scale assumptions such as strictly increasing or strictly decreasing returns to scale. However, from a methodological and empirical point of view, the estimation of α remains a question that must be answered. This paper proposes an effective methodology to estimate an optimal value of α based upon a goodness-of-fit strategy. A global method using a grid-search is presented first. In addition, for generalized Free Disposal Hull technologies, a minimum extrapolation principle is developed to estimate directly the optimal α-returns from a linear program. This study examined 63 US industries over the period 1987-2012 to demonstrate the relevancy of our approach.

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Keywords: α-Returns to Scale, Data Envelopment Analysis, Increasing Returns to Scale, Efficiency.

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1 Introduction

Boussemart, Briec, Peypoch, and Tavera (2009) introduced a specification of strictly increasing and decreasing returns to scale in multioutput technologies based on the notion of α-returns to scale. Their work relies on previous literature on homogeneous multioutput technology developed by Lau (1978) and Färe and Mitchell (1993). α-returns to scale model appears to be more theoretically founded than traditional Data Envelopment Analysis (DEA) returns to scale models in particular the non-decreasing returns to scale (NDRS) model. The α-returns explicitly imposes that the origin belongs to the production possibility set and is a proper way to model global increasing returns to scale as they are defined in the traditional economic literature.

While Boussemart et al. (2009) showed how these concepts can be implemented in a DEA framework using a variation of the piecewise homogeneous constant elasticity of substitution-constant elasticity of transformation (CES-CET) model introduced by Färe, Grosskopf, and Njinkeu (1988), they did not discuss the choice of the value for α. In the Boussemart et al. (2009) paper, α is an exogenous parameter as returns to scale (RTS) are fixed ex ante in traditional DEA models. In other words, assumptions of constant returns to scale (CRS), non-increasing returns to scale (NIRS) or NDRS are first imposed as axioms to define the technology, and the inefficiency scores are then calculated. In the same line, in Boussemart et al. (2009) the value of α has to be fixed first before computing inefficiency on the transformed data under the CES-CET specification.[1]

The main contribution of this study is that it considers α as an endogenous parameter and to estimate its optimal value as it is done in a traditional econometric framework. By not imposing a fixed α, we allow for a very flexible range of technologies in comparison to the three DEA cases (NIRS, CRS, NDRS). To make the comparison with Cobb-Douglas technologies, the RTS is defined as the sum of the output/input elasticities, which are free parameters in the specified technology. They are estimated from the observed data using least-square or maximum likelihood criteria. Our objective is also to estimate α from an observed set of data by maximizing a goodness-of-fit criterion that we define as the geometric mean of the efficiency scores.

Under a piecewise homogeneous CES-CET technology, a global approach using a grid-search method is proposed. While this strategy is quite general, it is also time-consuming, which is a well-known drawback. In addition, the α parameter is not really endogenous in the sense that it is selected as the best value within a predetermined set. Then a more promising approach which fully endogenizes α is developed. It is based on a minimum extrapolation principle for a generalized free disposal hull (FDH) technology defined as the

[1] One notable exception is the variable returns to scale (VRS) model in DEA for which none of the above RTS assumptions is imposed. However, we cannot directly compare α-returns, which imposes global RTS, to the VRS model, which is based on local RTS.

union of individual FDH technologies on which α-returns are applied. Finally, we show how the optimal α can be estimated by linear programming.

After presenting background information about of α-returns to scale in Section 2, the estimation methods are discussed in Sections 3 and 4. Section 5 discusses the tractability of the methodological framework by presenting an empirical implementation of the framework on 63 industries totaling the entire US economy during the period 1987-2012. Finally, the conclusions are presented in Section 6.

2 Background

2.1 Production Technology: Definition and Assumptions

The production technology transforms inputs $x = (x_1, \cdots, x_n) \in \mathbb{R}^n_+$ into outputs $y = (y_1, \cdots, y_p) \in \mathbb{R}^p_+$ under the technology T:

$$T = \left\{(x, y) \in \mathbb{R}^{n+p}_+ : x \text{ can produce } y\right\} \tag{2.1}$$

We suppose that the technology obeys the following axioms:

- T1: $(0, 0) \in T$, $(0, y) \in T \Rightarrow y = 0$ i.e., no free lunch;
- T2: the set $A(x) = \{(u, y) \in T : u \leq x\}$ of dominating observations is bounded $\forall x \in \mathbb{R}^n_+$, i.e., infinite outputs cannot be obtained from a finite input vector;
- T3: T is closed;
- T4: For all $(x, y) \in T$, and all $(u, v) \in \mathbb{R}^{n+p}_+$, we have $(x, -y) \leq (u, -v) \Rightarrow (u, v) \in T$ (free disposability of inputs and outputs).

2.2 α-Returns to Scale Technologies and Distance Functions

We first define the distance functions. The input Farrell measure is defined by $E_I(x, y) = \inf_\theta \{\theta \geq 0 : (\theta x, y) \in T\}$.

A production technology T is said to be *homogeneous of degree* α if for all $\eta > 0$:

$$(x, y) \in T \Rightarrow (\eta x, \eta^\alpha y) \in T. \tag{2.2}$$

Lau (1978) termed these technologies "almost homogeneous technologies of degree 1 and α" for all $\eta > 0$. A complete characterization is given by Färe and Mitchell (1993). Obviously, CRS corresponds to $\alpha = 1$ while strictly increasing returns corresponds to $\alpha > 1$ and strictly decreasing returns corresponds to $\alpha < 1$. Boussemart et. al. (2009) termed this property of the technology *α-returns to scale*. It has been shown in Boussemart, Briec, and Leleu (2010) have shown that under such an assumption almost all the existing measures (Farrell output measure, hyperbolic measure, proportional distance function) can be related in closed form under an α-returns to scale assumption.

We further propose a nonparametric model of production technologies for which distance functions can be calculated by solving the DEA models first introduced by Charnes, Cooper, and Rhodes (1978) for constant returns to scale and Banker, Charnes, and Cooper (1984) for variable returns to scale. Let us consider a set of J firms $A = \{(x_1, y_1), ..., (x_J, y_J)\} \in \mathbb{R}_+^{n+p}$. We denote $\mathcal{J} = \{1, \cdots, J\}$. The production technology can be estimated by enveloping the observed firms. Under this DEA framework, the production set for constant returns to scale is defined as:

$$T_{CRS} = \left\{ (x, y) \in \mathbb{R}_+^{n+p} : x \geq \sum_{j \in \mathcal{J}} \lambda_j x_j, y \leq \sum_{j \in \mathcal{J}} \lambda_j y_j, \lambda \geq 0 \right\} \tag{2.3}$$

We also use a more general *CES-CET* model introduced by Färe et al. (1988) and adapted by Boussemart et. al. (2009) to α-returns to scale. It consists of two parts: the output part, which is characterized by a *Constant Elasticity of Transformation* formula, and the input part, which is characterized by a *Constant Elasticity of Substitution* formula. Formally, we consider the map $z \mapsto z^r = (z_1^r, \cdots, z_m^r)$. For all $r > 0$, this function is an isomorphism from $\mathbb{R}_+^m$ to itself, and its reciprocal is the map $z \mapsto z^{1/r} = (z_1^{1/r}, \cdots, z_m^{1/r})$. Let us consider the following set:

$$T_{\gamma,\delta} = \Big\{ (x, y) : x \geq \Big(\sum_{j \in \mathcal{J}} \lambda_j x_j^\gamma \Big)^{1/\gamma}, \, y \leq \Big(\sum_{j \in \mathcal{J}} \lambda_j y_j^\delta \Big)^{1/\delta}, \, \lambda \geq 0 \Big\} \tag{2.4}$$

$T_{\gamma,\delta}$ satisfies T1-T4. It is obvious that $T_{CRS} = T_{1,1}$.

For the sake of simplicity, we shall denote the technical efficiency measure as:

$$E_I(x, y; \gamma, \delta) = \min\{\theta \geq 0 : (\theta x, y) \in T_{\gamma,\delta}\}. \tag{2.5}$$

In the context of our model, we then obtain:

$$E_I(x, y; \gamma, \delta) = \min_{\theta, \lambda \geq 0} \Big\{ \theta : \theta x \geq \Big(\sum_{j \in \mathcal{J}} \lambda_j x_j^\gamma \Big)^{\frac{1}{\gamma}}, y \leq \Big(\sum_{j \in \mathcal{J}} \lambda_j y_j^\delta \Big)^{\frac{1}{\delta}} \Big\} \tag{2.6}$$

It is then easy to see that the Farrell input technical efficiency measure can be computed on $T_{\gamma,\delta}$ using linear programming. An elementary change in the variables yields:

$$\begin{aligned} \big[E_I(x,y)\big]^{\gamma} = \min\ &\mu \\ s.t.\ \mu x^{\gamma} &\geq \sum_{j=1}^{n} \lambda_j x_j^{\gamma} \\ y^{\delta} &\leq \sum_{j=1}^{n} \lambda_j y_j^{\delta} \\ \mu, \lambda &\geq 0. \end{aligned} \tag{2.7}$$

3 General Procedure

Intuitively, the goodness-of-fit is an index that measures how far the observed data is closed from the production frontier. Obviously for one particular decision-making unit (DMU) the efficiency score is a goodness-of-fit index. Therefore, at the sample level we are looking for a pair of parameters (γ, δ) that maximize the input efficiency scores for all DMUs. A natural approach consists in considering the geometric mean of the Farrell input measures. The program one should solve is:

$$\max_{\gamma,\delta} L(X, Y; \gamma, \delta) \tag{3.1}$$

where

$$L(X, Y; \gamma, \delta) = \prod_{k \in \mathcal{J}} E_I(x_k, y_k; \gamma, \delta) \tag{3.2}$$

The problem we are facing is that even though each $E_I(x_k, y_k; \gamma, \delta)$ can be computed by linear programming, the global optimization program involves some nonlinear transformations of the observed data. Hence, a general procedure consists in the elaboration of a grid search method based upon a suitable discretization of two defined sets in which γ and δ are, respectively, assumed to lie. Let us consider $\underline{\gamma}, \overline{\gamma}, \underline{\delta}, \overline{\delta} > 0$ where $\underline{\gamma} < \overline{\gamma}$ and $\underline{\delta} < \overline{\delta}$. Assuming that $\overline{\gamma}$ and $\overline{\delta}$ are sufficiently large, we propose a discretization. This is accomplished by fixing $\underline{\gamma} = \gamma_0$, $\overline{\gamma} = \gamma_m$, $\underline{\delta} = \delta_0$, $\overline{\delta} = \delta_m$. More generally, one assumes that $\gamma_h = \gamma_0 + \frac{h}{m}(\gamma_m - \gamma_0)$ and $\delta_l = \delta_0 + \frac{l}{m}(\delta_m - \delta_0)$ for $h, l = 0, 1, ..., m$. Let us denote $\Gamma_m = \{\gamma_0, ..., \gamma_m\}$ and $\Delta_m = \{\delta_0, ..., \delta_m\}$. One needs to find the pair (γ_h, δ_l) maximizing the product of the efficiency scores. Hence, we are seeking to solve the maximization:

$$\max_{h,l} \prod_{k \in \mathcal{J}} E_I(x_k, y_k; \gamma_h, \delta_l). \tag{3.3}$$

If (h^*, l^*) yields a maximum, then the best approximation is given by the subset T_{γ^*,δ^*}.

Example 3.1 *Suppose that $n = p = 1$ and consider the following production units:*

	1	2	3
Input	1	4	5/2
Output	1	2	3/2

It is easy to see that the best approximation satisfying T1-T4 yields the production set:

$$T = \{(x, y) \in \mathbb{R}^2_+ : y \le \sqrt{x}\}.$$

This two-dimensional production set satisfies the $\frac{1}{2}$-returns to scale assumption. In particular, notice that $(x_2, y_2) = (4, 2) = (4.1, (4)^{\frac{1}{2}}.1)$. Moreover, since the production function $f : x \mapsto \sqrt{x}$ is strictly concave, $(x_3, y_3) = (5/2, 3/2) = (\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2})$ does not belong to the efficient frontier of T.

Fixing the value of $\gamma_h = \gamma_0 + 1/2m$ and of $\delta_h = \delta_0 + 1/2m$ such that $m = \{0, 1, 3, 4, 5, 6, 7, 8, 9, 10\}$ and $\gamma_0 = \delta_0 = 1/2$, the technology set becomes:

$$T^k_{1/2,1/2} = \{(x_k, y_k) : x_k \ge \sum_{j=1}^{3} (\lambda_j x_j^{1/2})^2, y_k \le (\sum_{j=1}^{3} \lambda_j y_j^{1/2})^2, \lambda \ge 0\}$$

$$\vdots$$

$$T^k_{3,1/2} = \{(x_k, y_k) : x_k \ge \sum_{j=1}^{3} (\lambda_j x_j^3)^{1/3}, y_k \le (\sum_{j=1}^{3} \lambda_j y_j^{1/2})^2, \lambda \ge 0\}$$

$$\vdots$$

$$T^k_{5,5} = \{(x_k, y_k) : x_k \ge \sum_{j=1}^{3} (\lambda_j x_j^5)^{1/5}, y_k \le (\sum_{j=1}^{3} \lambda_j y_j^5)^{1/5}, \lambda \ge 0\}.$$

And the input-Farrell technical efficiency is defined by:

$$(E^I_k)^{1/2}(x_k, y_k; 1/2, 1/2) = \min\{\theta^{1/2} : \theta^{1/2} x_k^{1/2} \ge \sum_{j=1}^{3} \lambda_j x_j^{1/2}, y_k^{1/2} \le \sum_{j=1}^{3} \lambda_j y_j^{1/2}, \lambda \ge 0\}$$

$$\vdots$$

$$(E^I_k)^3(x_k, y_k; 3, 1/2) = \min\{\theta^3 : \theta^3 x_k^3 \ge \sum_{j=1}^{3} \lambda_j x_j^3, y_k^{1/2} \le \sum_{j=1}^{3} \lambda_j y_j^{1/2}, \lambda \ge 0\}$$

$$\vdots$$

$$(E^I_k)^5(x_k, y_k; 5, 5) = \min\{\theta^5 : \theta^5 x_k^5 \ge \sum_{j=1}^{3} \lambda_j x_j^5, y_k^5 \le \sum_{j=1}^{3} \lambda_j y_j^5, \lambda \ge 0\}.$$

The computation is made through the following linear program:

$$
\begin{aligned}
(E_k^I)^\gamma(x_k, y_k; \gamma, \delta) = \min \quad & \theta^\gamma \\
s.t \qquad & \theta^\gamma x_k^\gamma \geq \sum_{j=1}^{3} \lambda_j x_j^\gamma \\
& y_k^\delta \leq \sum_{j=1}^{3} \lambda_j y_j^\delta \\
& \theta, \lambda \geq 0.
\end{aligned}
$$

The grid search is obtained through the computation of $L(X, Y; \gamma, \delta) = \prod_{k=1}^{3} E_k^I(x_k, y_k; \gamma, \delta)$*:*

Table 1: Grid search of the optimum values of γ and δ

δ \ γ	0.5	1	1.5	2	2.5	3	3.5	4	4.5	5
0.5	0.3	0.17	0.14	0.13	0.12	0.12	0.12	0.11	0.11	0.11
1	0.9*	0.3	0.21	0.17	0.16	0.14	0.14	0.13	0.13	0.12
1.5	0.34	0.52	0.3	0.23	0.19	0.17	0.16	0.15	0.14	0.14
2	0.13	0.9*	0.43	0.3	0.24	0.21	0.19	0.17	0.16	0.16
2.5	0.05	0.55	0.62	0.39	0.3	0.25	0.22	0.2	0.18	0.17
3	0.02	0.34	0.9*	0.52	0.37	0.3	0.26	0.23	0.21	0.19
3.5	0.01	0.21	0.65	0.68	0.47	0.36	0.3	0.26	0.24	0.22
4	0	0.13	0.47	0.9*	0.58	0.43	0.35	0.3	0.27	0.24
4.5	0	0.08	0.34	0.7	0.72	0.52	0.41	0.34	0.3	0.27
5	0	0.05	0.24	0.55	0.9*	0.62	0.48	0.39	0.34	0.3

(∗) the maximum value of L.

The maximum value of L *is* 0.9 *for the following pair of parameters:*

γ	0.5	1	1.5	2	2.5
δ	1	2	3	4	5

Since $\alpha = \gamma/\delta$*, it is obvious that* $\alpha^\star = 1/2$*.*

4 Minimal Extrapolation and Linear Programs to Estimate α

In this section, we propose a tractable procedure to find an optimal value of α under a generalized FDH technology. This approach fully endogenizes

α and estimates its value by linear programming. For each firm $k \in \mathcal{J}$, we consider an individual technology defined by:

$$Q_{\gamma,\delta}(x_k, y_k) = \Big\{(x, y) \in \mathbb{R}_+^{n+p} : x \geq \lambda^{1/\gamma} x_k, y \leq \lambda^{1/\delta} y_k, \lambda \geq 0\Big\}. \tag{4.1}$$

For each $j \in \mathcal{J}$ let us denote:

$$E_I^{(k)}(x_j, y_j; \gamma, \delta) = \min\{\theta : (\theta x_j, y_j) \in Q_{\gamma,\delta}(x_k, y_k)\} \tag{4.2}$$

By definition, one has $E_I^{(k)}(x_k, y_k; \gamma, \delta) = 1$. From Boussemart et. al. (2009) one can show that:

$$E_I^{(k)}(x_j, y_j; \gamma, \delta) = \Big[\max_{h \in car(y_k)} \frac{y_{j,h}}{y_{k,h}}\Big]^{\delta/\gamma} . \Big[\max_{i \in car(x)} \frac{x_{k,i}}{x_{j,i}}\Big]. \tag{4.3}$$

Suppose now that we define a global technology as the union of each individual technology by:

$$T^\dagger_{\gamma,\delta} = \bigcup_{k \in \mathcal{J}} Q_{\gamma,\delta}(x_k, y_k). \tag{4.4}$$

From Bousemart et. al. (2009) and using (4.2) one has:

$$E_I^\dagger(x_j, y_j; \gamma, \delta) = \min\{\theta : (\theta x_j, y_j) \in T^\dagger_{\gamma,\delta}\} \tag{4.5}$$

$$= \min_{k \in \mathcal{J}} \Big(\Big[\max_{h \in car(y_h)} \frac{y_{j,k}}{y_{k,h}}\Big]^{\delta/\gamma} . \Big[\max_{i \in car(x)} \frac{x_{k,i}}{x_{j,i}}\Big]\Big) \tag{4.6}$$

By defining $\alpha = \delta/\gamma$, the problem we are facing is to find $\alpha^\star$, which maximizes the quantity M defining the index of goodness of fit.

$$M(X, Y; \gamma, \delta) = \prod_{j \in \mathcal{J}} E_I^\dagger(x_j, y_j; \gamma, \delta) \tag{4.7}$$

subject to the constraint that $(x_j, y_j) \in T^\dagger_{\gamma,\delta}$ for all $j \in \mathcal{J}$.

By construction, for all $j \in \mathcal{J}$ one has $(x_j, y_j) \in T^\dagger_{\gamma,\delta}$, since the subset $T^\dagger_{\gamma,\delta}$ is constructed as the finite union of several individual production sets, each containing the observed production vectors.

It follows that this program can be solved by solving the maximization problem:

$$\max_{\alpha} \quad \prod_{j \in \mathcal{J}} \min_{k \in \mathcal{J}} \Big(\Big[\max_{h \in car(y_k)} \frac{y_{j,h}}{y_{k,h}}\Big]^{1/\alpha} . \Big[\max_{i \in car(x)} \frac{x_{k,i}}{x_{j,i}}\Big]\Big). \tag{4.8}$$

Taking, the logarithm, we obtain:

$$\max_{\alpha} \quad \sum_{j \in \mathcal{J}} \min_{k \in \mathcal{J}} \Big(\Big[1/\alpha \ln\Big(\max_{h \in car(y_k)} \frac{y_{j,h}}{y_{k,h}}\Big)\Big] + \ln\Big(\Big[\max_{i \in car(x)} \frac{x_{k,i}}{x_{j,i}}\Big]\Big)\Big). \tag{4.9}$$

Let us denote $\beta = 1/\alpha$, $c_{j,k} = \ln\Big(\max_{h\in car(y_k)} \frac{y_{j,h}}{y_{k,h}}\Big)$, and $d_{j,k} = \ln\Big(\Big[\max_{i\in car(x)} \frac{x_{k,i}}{x_{j,i}}\Big]\Big)$. The above program can be converted to:

$$\max_{\beta} \quad \sum_{j\in\mathcal{J}} \min_{k\in\mathcal{J}} \left(\beta c_{j,k} + d_{j,k}\right). \tag{4.10}$$

which is solved by the following linear program:

$$\begin{aligned} \max_{\beta,\lambda_j} \quad & \sum_{j\in\mathcal{J}} \lambda_j \\ st. \quad & \lambda_j \le (\beta c_{j,k} + d_{j,k}) \quad k\in\mathcal{J}, \quad j\in\mathcal{J}. \end{aligned} \tag{4.11}$$

The linear program has $|\mathcal{J}|+1$ variables and $|\mathcal{J}|^2$ constraints. Finally, one obtains $\alpha^\star = [\beta^\star]^{-1}$. Below, we use the same example introduced in the preceding section to provide an intuitive illustration of the above results.

Example 4.1 *Suppose that $n = p = 1$ and considers the input-output points $(x_1, y_1) = (1,1)$, $(x_2, y_2) = (4,2)$ and $(x_3, y_3) = (5/2, 3/2)$. The minimum extrapolation yields the production set:*

$$T = \big\{(x,y)\in\mathbb{R}^2_+ : y \le \sqrt{x}\big\}.$$

which satisfies the $\frac{1}{2}$-returns to scale assumption. Let us construct the individual technologies for each observation. We have:

$$Q_{\gamma,\delta}(x_1,y_1) = Q_{\gamma,\delta}(1,1) = \Big\{(x,y)\in\mathbb{R}^2_+ : x \ge \lambda^{1/\gamma}.1, y \le \lambda^{1/\delta}.1, \lambda\ge 0\Big\}.$$

$$Q_{\gamma,\delta}(x_2,y_2) = Q_{\gamma,\delta}(4,2) = \Big\{(x,y)\in\mathbb{R}^2_+ : x \ge \lambda^{1/\gamma}.4, y \le \lambda^{1/\delta}.2, \lambda\ge 0\Big\}.$$

$$Q_{\gamma,\delta}(x_3,y_3) = Q_{\gamma,\delta}(\frac{5}{2},\frac{3}{2}) = \Big\{(x,y)\in\mathbb{R}^2_+ : x \ge \lambda^{1/\gamma}.\frac{5}{2}, y \le \lambda^{1/\delta}.\frac{3}{2}, \lambda\ge 0\Big\}.$$

The computation of efficiency measures yields:

$$E_I^{(1)}(1,1;\gamma,\delta) = 1, \quad E_I^{(2)}(1,1;\gamma,\delta) = \big(\frac{1}{2}\big)^{\frac{\gamma}{\delta}}.4, \quad E_I^{(3)}(1,1;\gamma,\delta) = \big(\frac{2}{3}\big)^{\frac{\gamma}{\delta}}.\frac{5}{2}$$

$$E_I^{(1)}(4,2;\gamma,\delta) = \big(2\big)^{\frac{\gamma}{\delta}}.\frac{1}{4}, \quad E_I^{(2)}(4,2;\gamma,\delta) = 1, \quad E_I^{(3)}(1,1;\gamma,\delta) = \big(\frac{4}{3}\big)^{\frac{\gamma}{\delta}}.\frac{5}{8}$$

$$E_I^{(1)}(\frac{5}{2},\frac{3}{2};\gamma,\delta) = \big(\frac{3}{2}\big)^{\frac{\gamma}{\delta}}.\frac{2}{5}, \quad E_I^{(2)}(\frac{5}{2},\frac{3}{2};\gamma,\delta) = \big(\frac{3}{4}\big)^{\frac{\gamma}{\delta}}.\frac{8}{5}, \quad E_I^{(3)}(\frac{5}{2},\frac{3}{2};\gamma,\delta) = 1$$

For the entire technology, we have:

$$E_I(1,1;\gamma,\delta) = \min\Big\{1, 4\big(\frac{1}{2}\big)^{\frac{\gamma}{\delta}}, \frac{5}{2}\big(\frac{2}{3}\big)^{\frac{\gamma}{\delta}}\Big\}$$

$$E_I(4,2;\gamma,\delta) = \min\left\{\frac{1}{4}(2)^{\frac{\gamma}{\delta}}, 1, \frac{5}{8}\left(\frac{4}{3}\right)^{\frac{\gamma}{\delta}}\right\}$$

$$E_I(\frac{5}{2},\frac{3}{2};\gamma,\delta) = \min\left\{\frac{2}{5}\left(\frac{3}{2}\right)^{\frac{\gamma}{\delta}}, \frac{8}{5}\left(\frac{3}{4}\right)^{\frac{\gamma}{\delta}}, 1\right\}$$

Setting $\beta = \frac{\gamma}{\delta}$, we should solve the maximization program:

$$\max_{\beta}\Big(\min\left\{1, 4\left(\frac{1}{2}\right)^{\beta}, \frac{5}{2}\left(\frac{2}{3}\right)^{\beta}\right\}, \min\left\{\frac{1}{4}(2)^{\beta}, 1, \frac{5}{8}\left(\frac{4}{3}\right)^{\beta}\right\}, \min\left\{\frac{2}{5}\left(\frac{3}{2}\right)^{\beta}, \frac{8}{5}\left(\frac{3}{4}\right)^{\beta}, 1\right\}\Big)$$

Taking the logarithm and applying a maxi-min prodedure yields:

$$\begin{aligned}
\max_{\lambda,\beta}\ & \lambda_1 + \lambda_2 + \lambda_3 \\
& \lambda_1 \leq 0 \\
& \lambda_1 \leq \ln 4 - \beta \ln 2 \\
& \lambda_1 \leq \ln\frac{5}{2} + \beta\ln\frac{2}{3} \\
& \lambda_2 \leq -\ln 4 + \beta \ln 2 \\
& \lambda_2 \leq 0 \\
& \lambda_2 \leq \ln\frac{5}{8} + \beta\ln\frac{4}{3} \\
& \lambda_3 \leq -\ln\frac{5}{2} + \beta\ln\frac{3}{2} \\
& \lambda_3 \leq \ln\frac{8}{5} + \beta\ln\frac{3}{4} \\
& \lambda_3 \leq 0
\end{aligned}$$

The solution is $\beta^{\star} = \frac{\ln 4}{\ln 2} = 2$. Hence, we retrieve $\alpha^{\star} = \frac{1}{2}$.

5 An Analysis of the α-Returns to Scale for US Industries

This section discusses an empirical implementation of the minimum extrapolation principle to estimate the α-returns to scale. We applied our framework to 63 industries representing the entire US economy from the period 1987-2012.

5.1 The data

The underlying technologies are defined with one output and three inputs. The output is measured by the gross output while the input vector contains intermediate inputs, labor, and capital services (including equipment, structure, and intellectual property products). All basic data is collected from the production accounts established by the Bureau of Economic Analysis (BEA). For 63 different sectors (detailed in the Appendix) over the period 1987-2012, the variables are expressed in constants as US dollars (quantity indexes are weighted by their corresponding value levels in 2009). Volumes of capital services are estimated by the capital depreciation at the constant price. The quantity index of labor is based on the evolution of full-time equivalent employees.

5.2 Empirical implementation of the global model

The gross output, intermediate inputs, labor, and capital services for industry j at year t are denoted by $Y_j^t, M_j^t, L_j^t, K_j^t$, t. According to equation 4.11, the global α-returns to scale of year t can be computed by the following linear program:

$$\begin{aligned} \max_{\lambda_j^t,\beta^t} \quad & \sum_{j=1}^{63} \lambda_j^t \\ s.t \quad & \lambda_j^t \leq \beta^t c_{j,k}^t + d_{j,k}^t \\ & \forall k \in \{1,...,63\}, \quad \forall j \in \{1,...,63\} \end{aligned} \tag{5.1}$$

where $c_{j,k}^t = \ln\left(\frac{Y_j^t}{Y_k^t}\right), \quad d_{j,k}^t = \ln\left(\max\left[\frac{M_j^t}{M_k^t}, \ \frac{L_j^t}{L_k^t}, \ \frac{K_j^t}{K_k^t}\right]\right)$.

Running LP (5.1) for all $t \in \{1987, ..., 2012\}$, one can estimate both the time-series of efficiency scores and the α-returns to scale for the US global technology. More precisely, trough the value of the objective function, one can derive the efficiency scores equal to $\exp\left(\sum_{j=1}^{63} \frac{\lambda_j^t}{63}\right)$. The α-returns to scale are evaluated with the optimal levels of the variables β^t such as $\alpha^t = (\beta^t)^{-1}$.

5.3 Results

Table 2 presents the results for the efficiency scores and the optimal α-returns to scale.

Figure 1 presents the efficiency scores for the global technology over the period 1987-2012. According to our results, the US economy has continuously improved its productive performance for the last 26 years. The subprime crisis did not affect significantly the growth rate of efficiency scores. Moreover, the progress observed during the final years 2007-2012 reflects a continuation of the trend detected over the pre-crisis period (2003-2007).

Table 2: Technical efficiency for the US economy and returns to scale

Year	Efficiency	α_t	Year	Efficiency	α_t
1987	0.839	1.09	2000	0.875	1.05
1988	0.845	1.11	2001	0.871	1.04
1989	0.850	1.10	2002	0.885	1.03
1990	0.849	1.09	2003	0.883	1.02
1991	0.858	1.09	2004	0.899	1.01
1992	0.853	1.11	2005	0.915	1.02
1993	0.864	1.10	2006	0.918	1.01
1994	0.871	1.12	2007	0.914	1.03
1995	0.873	1.10	2008	0.920	1.01
1996	0.874	1.09	2009	0.923	1.02
1997	0.880	1.06	2010	0.937	1.01
1998	0.871	1.06	2011	0.938	1.00
1999	0.878	1.05	2012	0.944	1.01

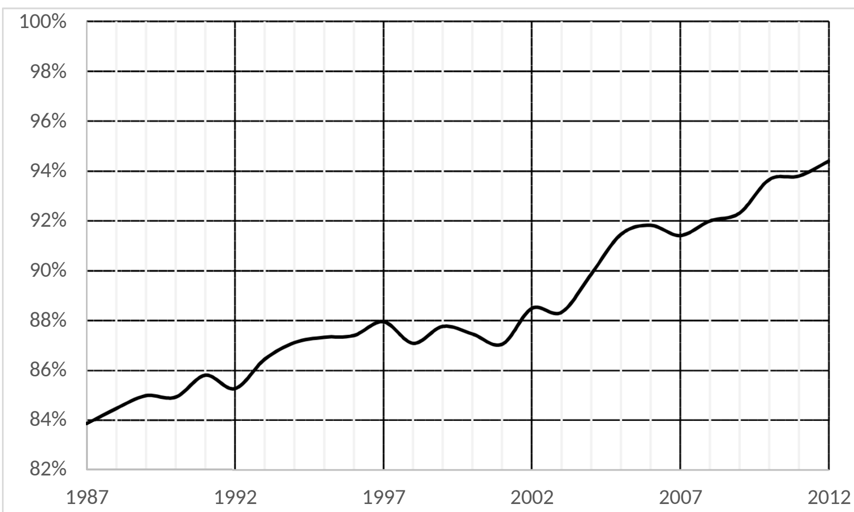

Figure 1: Efficiency scores of the US economy

Figure 2 details the levels and evolutions of the returns to scale. After an initial sub-period of nine years (1987-1995) during which US global technology was characterized by increasing returns to scale ($\alpha > 1$), another nine-year interval (1996-2004) shows a rapid decline.

Since that time, it appears that the US economy has nearly converged to a constant returns to scale technology. This implies that industries tend to their most productive scale size (MPSS), thereby improving their productive performances. This last result is in agreement with the positive evolution of the efficiency scores, which is confirmed by the strong negative correlation

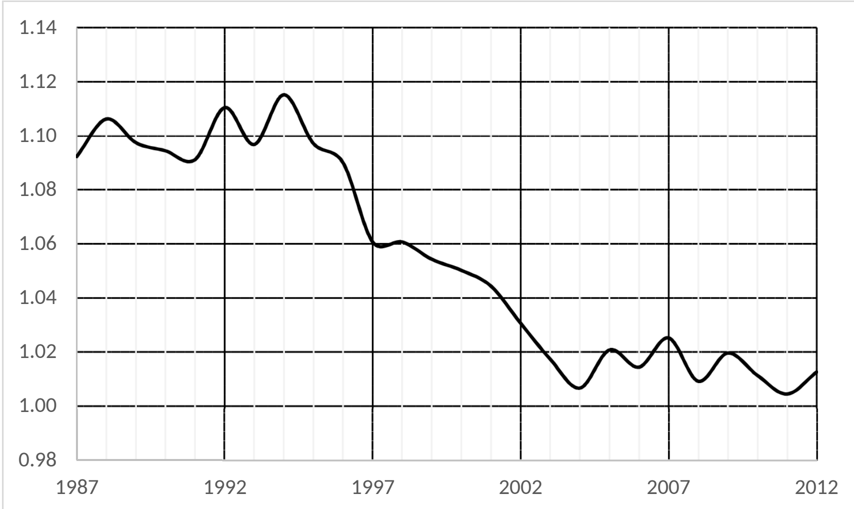

Figure 2: α-returns to scale for global technology

between the efficiency scores and the α-returns (R=-0.86).

Conclusion

While α was settled as an exogenous parameter in the initial contribution of Boussemart et al. (2009), the main contribution of this paper is to consider α as an endogenous parameter. We proposed a practical methodology to estimate its optimal value based upon a goodness-of-fit strategy. For generalized FDH technologies, we showed that the estimation of α can be done by linear programming. The empirical application to US industries shows the relevancy of the estimation strategy. In this paper, we considered a unique α that can be applied uniformly to all components of input/output-vectors and to all DMUs. In the future, this topic could be further studied by examining two additional ideas. First, it would be interesting to investigate a specific α for each input/output component. Second the idea of individual α-returns to scale can be considered where α could be specific to each DMU.

Appendix

J	Industry
1	Farms
2	Forestry, fishing, and related activities
3	Oil and gas extraction
4	Mining, except oil and gas
5	Support activities for mining
6	Utilities
7	Construction
8	Wood products
9	Nonmetallic mineral products
10	Primary metals
11	Fabricated metal products
12	Machinery
13	Computer and electronic products
14	Electrical equipment, appliances, and components
15	Motor vehicles, bodies and trailers, and parts
16	Other transportation equipment
17	Furniture and related products
18	Miscellaneous manufacturing
19	Food and beverage and tobacco products
20	Textile mills and textile product mills
21	Apparel and leather and allied products
22	Paper products
23	Printing and related support activities
24	Petroleum and coal products
25	Chemical products
26	Plastics and rubber products
27	Wholesale trade
28	Retail trade
29	Air transportation
30	Rail transportation
31	Water transportation
32	Truck transportation
33	Transit and ground passenger transportation
34	Pipeline transportation
35	Other transportation and support activities
36	Warehousing and storage
37	Publishing industries (includes software)
38	Motion picture and sound recording industries
39	Broadcasting and telecommunications
40	Information and data processing services
41	Federal Reserve banks, credit intermediation, and related activities
42	Securities, commodity contracts, and investments
43	Insurance carriers and related activities
44	Funds, trusts, and other financial vehicles
45	Real estate
46	Rental and leasing services and lessors of intangible assets
47	Legal services
48	Computer systems design and related services
49	Miscellaneous professional, scientific, and technical services
50	Management of companies and enterprises
51	Administrative and support services
52	Waste management and remediation services
53	Educational services
54	Ambulatory health care services
55	Hospitals and nursing and residential care facilities
56	Social assistance
57	Performing arts, spectator sports, museums, and related activities
58	Amusements, gambling, and recreation industries
59	Accommodation
60	Food services and drinking places
61	Other services, except government
62	Federal government
63	State and local government

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