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Abstract

This paper examines the definition of a technology and the choice of a model orientation prior to the analysis of portfolios of financial assets with Data Envelopment Analysis. We acknowledge the previous contributions in the field and provide answers to the questions raised in Cook, Tone & Zhu (2014). These answers allow to determine the purpose of the study and to define the underlying 'financial' technology through the identification of the decision-making units and the selection of input and output variables in a multi-moment framework. We also show their impact on the traditional set of axioms that further characterizes the technology and propose some adjustments to the traditional models. We provide illustrations to show the effects of such changes on the scores of technical efficiency and ranking of the portfolios.

Keywords : Data Envelopment Analysis; Portfolio Frontier; Model orientation; Mean-Variance; Risk preferences

Portfolio analysis with DEA: prior to choosing a model

1. Introduction

Since the seminal work of Markowitz (1952) on portfolio selection, several tools, models and approaches for decision-making have been developed in the financial and economics literature to evaluate the performance of portfolios of financial assets. The mean-variance approach introduced by Markowitz relies on the construction of a frontier relative to which portfolio performance is measured. Parallel to this literature, a methodology for performance measurement of decision-making units (DMUs) was being developed in the economics and operational research literature with Data Envelopment Analysis (DEA), a non-parametric tool. The junction between portfolio selection through quadratic optimization and the methodology with DEA inherited from operational research can be found in Sengupta (1989), but it took until Murthi, Choi & Desai (1997) to identify DEA as an "*extremely useful technique for measuring efficiency*" of mutual funds. While they used a CCR [1] model on mutual funds, the following contribution of McMullen & Strong (1998) used a BCC [2] model. Premachandra, Powell & Shi (1998) then introduced stochastic DEA and studied stock indexes, and Morey & Morey (1999) used DEA for multi-horizon portfolio analysis. Since then, numerous studies have strictly transposed the whole methodology used in production theory and operational research to the study of portfolios of financial assets with DEA without necessarily questioning the accuracy of such transposal. Though these works contributed to the elaboration of a general approach for measuring single-period portfolio efficiency in multi-moments frameworks (see Briec & Kerstens, 2010), some adjustments to the traditional methodology can still be proposed in order to make it suited to the analysis of financial assets, by so much as the definition of the underlying technology or the choice of a model orientation.

In a recent article Cook, Tone & Zhu (2014) list several modeling issues raised by an ill-adapted transposal of DEA models to various fields of research. In order to bring adequate solutions to these issues and ensure proper modeling, they also list a series of questions that should be answered prior to any analysis with DEA. In this article we intend to question the traditional definition of a technology in the context of portfolio analysis, realized through the identification of DMUs, the proper selection of input and output variables and the definition of a set of axioms. Answering the questions raised in Cook, Tone & Zhu (2014), we identify what can be the purpose of performance measurement and analysis and how it can impact the identification of DMUs or model orientation. We propose to apply a similar treatment to risk as the one used for byproducts in weakly disposable DEA models, provide arguments to support our choice and show the consequences on the definition of the technology and model orientations. We eventually propose to modify the traditional set of axioms inherited from production theory to take into account the correlations between assets' returns, the possibility of risk-free investments and the implications of risk reduction on the level of expected return.

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[1] As introduced in Charnes, Cooper & Rhodes (1978)

[2] As introduced in Banker, Charnes & Cooper (1984)

2. Definition of a financial technology prior to the analysis

2.1. "What is the purpose of the performance measurement and analysis?"

The first question raised by Cook, Tone & Zhu (2014) relates to the purpose of performance measurement and analysis. In both portfolio theory and production theory, performance evaluation can be considered under the two complementary angles of technical and allocative efficiency. Technical efficiency of financial assets can provide information on the return on investment relative to the various costs incurred, independently from any system of prices (with no regards to the decision-makers' preferences in portfolio analysis). Allocative efficiency is estimated relative to a profit-maximizing strategy (relative to a utility-maximizing strategy in portfolio analysis, provided that the parameters of the utility function are known). While most studies in the literature focus on technical efficiency, Briec, Kerstens & Lesourd (2004) and Briec, Kerstens & Jokund (2007) also show how economic efficiency can be reached.

The purpose of the study can then range from portfolio selection (from the perspective of investors), portfolio construction or portfolio management (either from the perspective of individual investors or fund managers) to efficiency measurement of the financial markets. While technical efficiency of portfolio construction can be measured relative to the set of the portfolio's possible holdings (individual securities), technical efficiency of portfolio selection ought to be measured relative to a set of already constituted portfolios. [3] In this latter case, convexity will not be assumed as a regularity condition; the objective of the study consists in benchmarking existing funds and either provide a ranking of the portfolios (as in Premachandra, Powell & Shi (1998) or Morey & Morey, 1999) or simply study the technically inefficient portfolios (as in Basso & Funari, 2001). Further analysis can then be made on the determinants of portfolio inefficiency in order to study the drivers of the funds' performance in two-steps DEA (as in Galagedera & Silvapulle, 2002), which should theoretically converge to the results of a fundamental analysis. Allocative efficiency of portfolio management or portfolio selection can also be measured in order to assess to which extend fund managers or investors succeed in reaching their individual objectives regarding either the fund's orientation or their respective preferences towards risks. [4]

Corollary questions to the purpose of the study are the choice of a theoretical framework to study technical efficiency on the one hand, and the identification of the study-makers' or decision-makers' preferences to study allocative efficiency on the other hand. Regarding the theoretical framework, Cook, Tone & Zhu (2014) remind the importance of spending more time, prior to the analysis, determining what matters to the study-maker ("*the precise measures deemed important by management*"). The theoretical framework impacts the definition of the technology by determining the set of inputs, outputs and the definition of a set of regularity conditions. Though performance evaluation can accurately be based on past records, its corollary risk measurement may also require resorting to fundamental analysis if it has a predictive intent, so that expectations about future prices can be formed from accounting information and

[3] The literature on portfolio performance with DEA has focused until now on already constituted portfolios such as mutual funds, hedge funds or CTAs.

[4] Allocative efficiency of portfolio management for already constituted funds would then have to be considered from the perspective of the funds' managers, as the investors in a fund are not the decision-makers and their utility functions can differ substantially, as discussed in Ballestero & Pla-Santamaria (2004).

used in a decision-making process. If this is especially true for individual stocks, performance assessment and risk measurement of investment funds can still accurately rely on historical records on the grounds that higher performance demonstrates superior management skills that can back up more favorable expectations about the funds' future performance. Predicting price and return remains a difficult challenge; as a consequence, whenever the objective of the study relates to portfolio selection in order to achieve future performance, decision-makers deal with the records of past returns to form their expectations. Most studies on portfolio performance with DEA until now have consequently adopted a retrospective approach, though Briec & Kerstens (2009) introduce a few thoughts for a prospective approach.

Regarding the parameters of the utility functions, preference for higher mean returns and aversion to a higher variance of returns are systematically assumed by study-makers. However, a whole part of the literature in economics and empirical finance has developed around the question of riskier choices and preferences for increases in even moments (from Blum & Friend (1975) to Golee & Tamarkin (1998), Astebro (2003), or Bali, Cakici & Whitelaw, 2010). From this literature we keep here only one major finding: the choice of a random prospect with a higher variance does not necessarily imply a preference for an increase in risk [5] in a multi-moments framework. In other words, risk aversion and the choice of a portfolio of financial assets with a higher variance are not mutually exclusive. At constant mean, such choice can be attributed to the impact of higher-order moments; in any other case, it can simply result from a utility function that attributes higher utility to riskier prospects, provided that they offer a high enough return. For this reason, measuring performance relative to a set of efficient but systematically less risky DMUs is too restrictive and ought to be reconsidered.

2.2. "What are the decision-making units and the outputs and inputs to be used to characterize the performance of those DMUs?"

In portfolio analysis, DMUs can either be individual securities or portfolios of securities such as investment funds or indexes, depending on the object of the study. One specificity of such DMUs is the dependence of their prices or returns, as opposed to the independence between the activities of DMUs traditionally assumed in the various fields on which DEA has been applied. This dependence of DMUs' returns can result in a non-convex technology set. Cook, Tone & Zhu (2014) remind that any process assimilated to a production process has to be clearly understood prior to the selection of input and output variables and remind the importance of ensuring that they "*properly reflect, to the greatest extent possible, the "process" under study*". The choice of output variables has always been quite consensual in the literature that uses DEA for the assessment of portfolio performance. On the contrary, the multiplicity of measures that have been proposed to account for input variables shows that there is no consensus either on the theoretical framework to be used for the study, on the measures to be used to account for some risk metrics or on the input or output status attributed to various measures.

On the one hand, desirable outcomes have always been included in the set of output variables and the choice of measures of reward as outputs obtains a consensus. Over various measures of average return

[5] In the sense of Rotschild & Stiglitz (1970), meaning a preference for the application of a Mean-Preserving Spread.

proposed in the literature, one can find either mean or compounded return on past performance, [6] as well as expected return on future performance. Returns can either be expressed as gross or net returns [7] and as sometimes as excess return above the market's performance, either before or after tax. Minimum returns can also be found in some studies (see Wilkens & Zhu (2001), Glawischnig & Sommersguter-Reichmann, 2010) as well as the number of days/months with positive returns on a daily/monthly distribution of returns, (see Gregoriou & Zhu, 2007), upper (or higher) partial moments (see Gregoriou (2003), Gregoriou & al., 2005) or consecutive gains (see Gregoriou & Zhu, 2007). McMullen & Strong (1998) also take into consideration the returns over various time-horizons. Traditional performance indicators as the Sharpe and Treynor ratios, the Jensen's alpha or the reward-to-half-variance index (see Basso & Funari, 2005) have been considered as output measures, as well as other desirable outcomes such as a positive skewness of the distribution of returns (see Wilkens & Zhu, 2001), indicators of stochastic dominance or of the ethical orientation of a fund. In these latter cases a qualitative indicator can be added to the set of output variables (the ethical factor of Basso & Funari (2003) or the stochastic dominance indicator of Basso & Funari (2001, 2005) and Kuosmanen, 2005).

On the other hand, various costs associated to investment as well as undesirable outcomes have always been included in the set of input variables. Murthi, Choi & Desai (1997) consider standard deviation of the returns as an input variable, as well as the transaction costs that managers incur "*in order to generate the return*" such as an expense ratio (management fees, marketing and operational expenses), additional loads for some funds (sales charges, redemption fees) or management turnover. Similarly, McMullen & Strong (1998) consider standard deviation of returns, sales charges, expense ratio and minimum investment as inputs. Eling (2006) includes the minimum investment or the lock-up period in the set of input variables, as well as an indicator of trading activity or excess kurtosis. Basso & Funari (2001) propose various risk measures (standard deviation of returns, root of the half-variance or beta coefficient) and additional costs as input variables. Choi & Murthi (2001) also consider managerial skills, market and institutional factors in their set of input variables. Wilkens & Zhu (2001), Galagedera & Silvapulle (2002), Basso & Funari (2003) then used many of the above-listed measures on multiple time horizons. Glawischnig & Sommersguter-Reichmann (2010) introduced lower partial moments as new measures of risk and input variables. Eling (2006), Gregoriou & Zhu (2007) or Branda (2015) also use various drawdowns (maximum or average drawdown, standard deviation of drawdown, Value-at-Risk, conditional Value-at-Risk or Modified Value-at-Risk), the beta factor and residual volatility or tracking error, and Gregoriou & Zhu (2005) use the proportion of negative returns in a distribution of returns as an input.

For all contributions listed above risk measures are treated as input variables and return measures as output variables, which can be explained by two main reasons. On the one hand, decision-making in production is based on input reduction and output augmentation and decision-making in finance is traditionally based on risk reduction and return augmentation. On the other hand the frontier of efficient portfolios is similar in shape to a production frontier; the analogy has then been made for long between efficiency analysis in production and performance analysis in finance. This analogy and the desirability for

[6] Arithmetic means of the distribution of returns assume withdrawal of gains while geometric means of the distribution of (gross) returns assume reinvestment of past gains.

[7] What we refer to as the 'gross return' is similar to the 'capitalization factor' of Basso & Funari (2007).

return and commonly accepted undesirability of risk have led numerous authors to consider the risk-return relationship of financial assets as equivalent to an input-output relationship.

Several criteria that do not necessarily relate to the notion of a "production process" have also been either explicitly proposed or implicitly used for the selection of inputs and outputs in the literature that uses DEA on financial portfolios. One of them relates to the nature of the interaction between input and output variables, and more precisely whether it is characterized by a causal relationship of the production-kind or not. The identification of the causal relationship, if any, considerably eases the choice of input and output variables among the set of relevant variables. Another criterion relates to the behavior of decision-makers towards input or output variables inferred from what's assumed to be their preference or aversion to these variables. This second criterion is often considered as a mean to address the need to take into consideration investors' preferences, and more especially the measures that are relevant in their opinion. However, once the first criterion has been successfully applied, the second criterion becomes pointless. Still, the choice of a theoretical framework prior to the analysis should ensure consistency in the definition of any 'financial production process', determine its level of aggregation and drive the selection of input and output variables.

2.3. Resulting financial technology in a mean-variance framework

Under a mean-variance framework, defining the relationship between the level of second-order risk (measured by the variance or standard deviation of returns) and the realized return on investment as a 'production' relationship would lead to an erroneous and incomplete representation of the technology. On the one hand, no functional form can express the expectation of a higher return as a result of a riskier investment. On the other hand, the positivity of the risk-return relationship has been proven wrong on some categories of assets (on alternative investments for instance) or in case of the so-called 'leverage effect' (when the past returns are negatively correlated to the future volatility of some stocks, on short-term horizons). The risk-return relationship, traditionally considered positive, is consequently no appropriate support for the representation of the technology.

As the measures associated to risk and return both have the same source (the distribution of past returns), it then seems consistent to treat them all as outputs. [8] Similarly, higher moments of the distribution to be included in multi-moment framework (as skewness or kurtosis) and any measure characterizing the distribution of returns could be regarded as outputs. A similar understanding of the 'financial technology' can be found in Anderson & al. (2004) who consider that any benefit arising from an investment is an output and the investment itself is the input. However, while they consider that the level of risk is "taken" by the investor and is therefore part of the initial investment made in the portfolio, we consider that it cannot be quantified a priori and is therefore not 'taken' but rather generated simultaneously to the distribution of returns. A timing assumption also underlies any production process: output generation must be preceded by the supply of some input, as it results from the latter and the production process necessarily takes some time. This sequence is realized here: all outputs are generated simultaneously after the initial investment has been made.

[8] Liu, Zhou, Liu & Xiao (2015) provide a similar argument.

From the above specifications can be defined the financial technology in (2.3.1), assuming an input vector $\mathbf{x}$ of initial investment and an output vector $\mathbf{y} = (\boldsymbol{\mu}, \boldsymbol{\sigma}^2)$ of the first two moments of the distribution of returns on the initial investment.

(2.3.1) $P(\mathbf{x}) = \{(\boldsymbol{\mu}, \boldsymbol{\sigma}^2) : (\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\sigma}^2) \in T\}$ for all $\mathbf{x} \in \mathbb{R}_+^n$

The set of possible portfolios can be defined as the set of all possible combinations of shares q_j of initial investment in asset j. Let R_j be the return on asset j with $j \in J$ at time t. We consider R_j as a random variable defined on the probability space (Ω^j, F^j, P^j), with Ω^j the sample space (or set of all possible outcomes) of the variable R_j, $F^j = (F_1^j, F_2^j, \dots, F_E^j) = \{F_e^j : e \in E\}$ the set of events that can influence the outcomes of the variable R_j, with $E \in \mathbb{N}^*$ the number of possible events and $P^j = (p_1^j, p_2^j, \dots, p_E^j) = \{p_e^j : e \in E\}$ for all j the assignment of probabilities to every event contained in F^j. The set of possible portfolios (set of all possible combinations of proportions, or 'portfolio possibility set') can be defined as follows:

(2.3.2) $\Im = \{ \mathbf{q} \in \mathbb{R}^J : \sum_{j=1}^{J} q_j \leq 1 \ , \ q_j \geq 0 \text{ for all } j\}$[9] and $\Im \neq \emptyset$

As reminded in Lamb & Tee (2012) who consider the portfolio possibility set as the set of all random variables 'possible portfolio', from this portfolio possibility set, any measure is a function $g: \Im \to \mathbb{R}$, with the value of this measure being a population statistic (a sample statistic in practice) and a real number (rather than a random variable). It is always assumed that the portfolios' returns are realizations of the random variable. From the set of possible portfolios defined above can be defined any measure that characterizes the portfolios. Depending on the chosen theoretical framework, the representation of the set of possible portfolios is then expressed as the set of all the related measures such that $\mathbf{q} \in \Im$. The first two non-standardized moments that characterize the portfolios' distributions of returns are defined for any portfolio P in (2.3.3) and (2.3.4), with μ_i the mean return of a distribution of (net) returns on investment on DMU i, with q_i the share of DMU i in portfolio P, with σ_i the standard deviation of returns of a distribution of (net) returns on investment on DMU i and with ρ_{ij} the linear correlation coefficient between the distributions of (net) returns on DMUs i and j. The representation of the set of possible portfolios in the mean-variance framework can then be defined as the set of all mean-variance combinations of portfolios such that $\mathbf{q} \in \Im$, with GR_{MV} a non-convex set, as in (2.3.5).

(2.3.3) Mean (or expected) return: $E_P = \sum_{j=1}^{J} q_j \mu_j$

(2.3.4) Variance of returns: $V_P = \sum_{i=1}^{J} \sum_{j=1}^{J} q_i q_j \rho_{ij} \sigma_i \sigma_j$

(2.3.5) $GR_{MV} = \{ (E_P, V_P) : \mathbf{q} \in \Im \}$

[9] Following the treatment of short sales commonly accepted in the literature, short sales are treated like negative purchases, such that any negative q_j indicates that the asset j is sold short. If short sales are allowed, the share invested in each asset is no more constrained to be non-negative. A lower bound to q_j consequently determines whether or not short sales are allowed.

2.4. "What is the appropriate model orientation?"

Based on the literature on decision-makers' preferences in the expected utility framework, the following model orientations appear theoretically grounded in an output-oriented mean-variance model that assumes a utility function of mean and variance only.

(1) ***Return augmentation at constant variance***, as proposed in Morey & Morey (1999) is recommended whenever the parameters of the utility functions are unknown or if it is assumed that the tolerance of decision-makers for risk is equivalent to the evaluated DMU's variance. If the level of risk carried by the evaluated funds reflects the maximum level of risk that the funds' managers are willing to face, this orientation provides information on how much more return could have been generated for that level of risk. In no case is this orientation related to risk-neutrality. [10]

(2) ***Risk reduction at constant mean***, as proposed in Morey & Morey (1999), allows to measure to which extend fund managers succeed in reducing the risk of their portfolio at a given mean return – or investors in the evaluated portfolios succeed in selecting the less risky portfolio at a given level of required return. This orientation is accurate whenever the parameters of the utility function (consequently the coefficient of risk aversion) are unknown but risk aversion is assumed to be shared by all decision-makers: under such conditions any portfolio with a lower variance will be preferred, at a constant mean.

(3) The ***simultaneous risk reduction and return augmentation*** (the Efficiency Improvement Possibility (EIP) function introduced in Briec, Kerstens & Lesourd (2004) and used in Briec & al. (2007) and Briec & Kerstens, 2009) can also be considered when the parameters of the utility function are known and allocative or economic efficiency is measured. However, if the parameters remains unknown, nothing can theoretically justify this orientation even though all DMUs with a lower variance and a higher return dominate the set of observed DMUs. Indeed, only the evaluation relative to a portfolio with a lower variance at a constant mean guarantees that the choice results from risk-aversion (risk-lovers could prefer a portfolio with a higher return and a lower variance for the higher return it provides). Unless the parameters of the utility function are known, nothing can theoretically justify a measurement of performance relative to a DMU that has neither the same level of risk nor the same level of return. [11] When the parameters remain unknown but risk aversion is a key assumption, we recommend to use the second model orientation, as evaluation relative to a DMU with lower variance implies risk aversion at constant mean only.

(4) A ***simultaneous augmentation of risk and return*** has not been considered so far in the literature with DEA on financial portfolios. Yet, this orientation is especially legitimate whenever it is assumed that the higher the expectation on return, the higher the level of risk investors are ready to take, even under the assumption of risk aversion. If risk and return are considered as outputs, it is simply an output-based Debreu-Farrell measure of technical efficiency. This radial measure, by keeping the ratio risk/return constant, guides the evaluated DMU towards the frontier along the 'expansion path', its output mix. As proved by Russell (1985), radial measures present several desirable properties especially when market prices – or preferences in our context are unknown. If expected future returns are used instead of the mean return on past records,

[10] Assuming risk neutrality would merely imply removing any risk measure from the model and evaluate the DMUs relative to the one that provides the highest return.

[11] It could for instance lead to measure the performance of a fund relative to another fund that has a different objective regarding risk management.

this expansion path is particularly relevant. If however mean return of historical records are used, this model orientation would be justified in cases where the R-squared of the regression is positive and significant, so that it can support the idea of a positive risk-return relationship on the market of the studied sample of portfolios. [12] This direction is also consistent with the assumption of jointness later introduced in this article. In a traditional production framework, one desirable property of the radial measure is that it provides information of the variation in revenue, independently of the value of prices, known or not (Russell, 1985). For portfolio analysis under the expected utility framework and for a utility function that can be expressed as a linear function of the first two moments as in (2.4.1), such direction can provide information on the variation in utility that results from reaching the efficient frontier, regardless of the parameters of the utility function. A radial expansion of the observed DMU (E_o, V_o) by a factor n results in (E_o^*, V_o^*) as in (2.4.2), and the utility U^* associated to this projection to the frontier can be expressed as in (2.4.3).

(2.4.1) $U(E,V) = \boldsymbol{\mu}E - \boldsymbol{\rho}V$ with $\boldsymbol{\mu} \geq 0$ and $\boldsymbol{\rho} > 0$ in case of risk-aversion and $\boldsymbol{\rho} < 0$ in case of a preference for risk

(2.4.2) $(E^*, V^*) = \big((1+n)E, (1+n)V\big)$

(2.4.3) $U_o^* = (1+n)U_o$ with $U_o^* = U(E_o^*, V_o^*) = \boldsymbol{\mu}(1+n)E_o - \boldsymbol{\rho}(1+n)V_o$

2.5. "What is an appropriate number of DMUs, given the number of inputs and outputs chosen?"

One additional issue to the identification of input and output variables that is often raised in the literature and also mentioned in Cook, Tone & Zhu (2014) relates to the appropriate number of DMUs to constitute a sample in an analysis with DEA. A reciprocal question relates to the maximum amount of variables to be allowed in the set of input and output variables, knowing that additional variables most often result in an increase in the number of efficient DMUs. This phenomenon is sometimes referred to as the *curse of dimensionality* and is especially crucial for non-parametric estimators (see Simar & Wilson, 2000).

This question can be addressed in two way. On a purely theoretical and statistical basis, one can study the influence of the number of DMUs and the number of input and output variables on the speed of convergence of the estimator (see Kneip, Park & Simar, 1998). On an empirical basis, numerous authors recommend restricting as much as possible the number of input and output and some rules of thumb relating the number of input and outputs to the number of DMUS have been proposed without being theoretically grounded. Empirical testing is also a way to deal with this issue. By varying the sample size and the number of inputs and outputs, we could test if efficiency scores are robust across different specifications. A recent contribution of Liu, Zhou, Liu & Xiao (2015) deals with how well various DEA models used in performance measurement of financial portfolios approximate the portfolio frontier.

[12] By way of example, for the data used in our illustration and taken from Morey & Morey (1999), the R^2 of the regression on 3-year returns is equal to 0.24 without any risk-free asset, to 0.4 when cash is included to the set of DMUs, and 0.37 when a risk-free asset with a monthly mean return of 0.3% is added to the set (used by the authors for the calculation of the Sharpe ratio). This R^2 decreases to 0.14 or 0.27 with the risk-free asset for the 5-year horizon, and to 0.09 and 0.31 for the 10-year horizon.

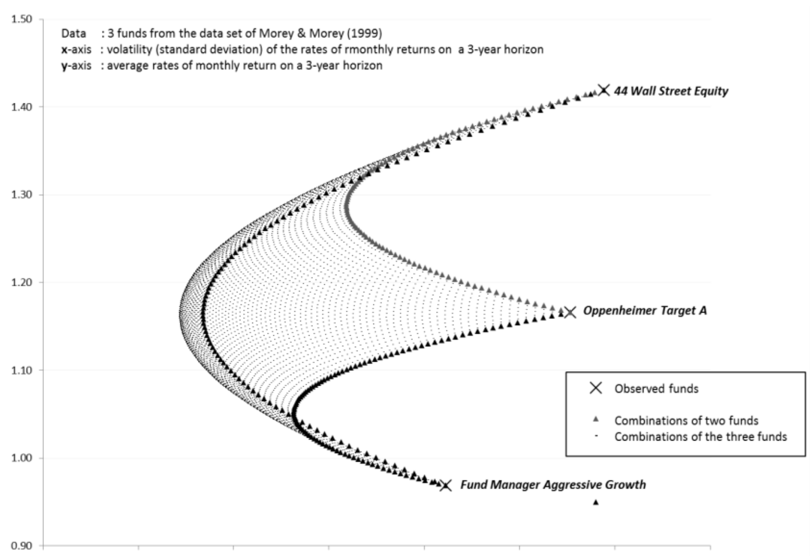
3. Modifications to the traditional set of axioms

3.1. Non-convexity of portfolio possibility sets

A mean-variance analysis implies working on the output correspondence of the production possibility set when risk is defined as an output, which ensures that the set is closed. Performance measurement relative to the frontier can then be made regardless of the system of preferences while any preference for risk couldn't be studied when risk was considered as an input. As a consequence of the choice of DMUs (already constituted portfolios) or the linear correlation between the assets' distributions, convexity cannot be imposed as regularity condition for a financial technology.

Traditional DEA in production relies on the notion of DMUs to which is associated a production plan (a vector of inputs produces a vector of outputs) and on the implicit assumption of independence between DMUs is made. Such hypothesis of independence cannot be made for financial assets for two main reasons. On the one hand, independence between production units could in no way be translated into independence between financial assets; it would instead be equivalent to an assumption of perfect linear correlation. On the other hand, linear independence and perfect linear correlation between financial assets are only particular cases that may never be observed in reality, or very seldom. The returns of financial instruments exhibit some covariance and therefore some linear correlation. The latter can be null when the distributions of returns do not co-vary, and is implied by the independence between variables, but the reciprocal is not true: independence is no way implied by a zero linear correlation. As illustrated in Figure 3.1 below, some degree of linear dependence between financial assets impacts the level of risk of any linear combination of financial assets. It can be reduced through diversification; still, even linear independence (a zero linear correlation) between DMUs would not anyway result in linear combinations of the initial DMUs' risk levels for some risk measures. The minimum level of risk when measured by the variance of a distribution of returns is a convex quadratic function of the mean return; the resulting non-convexity of the set frontier is consequently an issue only whenever we measure performance using a direction vector that follows an expansion path.

Figure 3.1 Feasible combinations of three funds in a mean-standard deviation space



In a mean-variance framework, the mean return of any portfolio of DMUs can be expressed as a linear function of the mean returns of the individual DMUs (assets) that compose the portfolio, as in (2.3.3). On the contrary, the variance of the portfolios' returns cannot be expressed as a linear function of the respective variances of returns of the DMUs that compose the portfolio, as in (2.3.4). In case of a perfect linear correlation ($\rho_{ij} = 1$) between the assets that compose the portfolio, diversification does not impact the portfolio's total risk. The portfolio variance is then equal to $\sigma_P^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} q_i q_j \sigma_i \sigma_j$. In case all assets in the portfolio were independent, the portfolio variance would be $\sigma_P^2 = \sum_{i=1}^{n} {q_i}^2 {\sigma_i}^2$ and diversification would impact (reduce) the total portfolio's risk. When variance is the measure of second-order risk, convex combinations of assets can then never result in linear combinations of both the assets' risk and return measures, which would ensure the convexity of the technology set in a mean-variance framework.

If standard deviation of returns – which can be considered as an equivalent measure to variance in terms of information – was the measure of second-order risk, linearity of the risk and return measures of the combinations and the assets could only be observed in the unrealistic case where all assets portfolio were perfectly linearly correlated ($\rho_{ij} = 1$), such that $\sigma_P = \sum_{j=1}^{n} q_i \sigma_i$ – or whenever a risk-free asset is included in the portfolio, such that $\sigma_P = \sum_{j=1}^{n-1} q_i \sigma_i$, which makes the optimization problem trivial. Linearity can then be observed only under particular conditions on the chosen measure of risk and correlation links between the assets. Lamb & Tee (2012) and Branda (2015) recently proposed to work with "diversification-consistent models" that solve this issue by using other risk measures than variance or standard deviation. As noticed by Lozano & Gutierrez (2008), all linear programming approaches that have been used to measure efficiency of mutual funds – except for Daraio & Simar (2006) – have overestimated risk by considering convex combinations of the DMUs' respective levels of risk to account for portfolio risk. For Brandouy, Kerstens & Van de Woestyne (2013) "*it could at best be considered a type of linear approximation of a possibly non-linear portfolio model*". While the traditional assumption of free disposability on the risk dimensions extends the feasible set to a convex set under the traditional approach that defines risk as an input, it does not make the output correspondence of the technology set convex when risk is defined as an output. Figure 3.1 provides an illustration of non-convexity on the output set.

3.2. Axioms of "no free lunch" and the possibility of inaction

The axiom of "no free lunch" has often been considered in financial analysis as equivalent to an assumption of fair pricing on the markets. However, Barberis & Thaler (2003) remind that this equivalence holds in efficient markets only, and while correct pricing implies no free lunch, the opposite is not true. Market inefficiency does not necessarily imply free lunches, and market inefficiency is certainly not to be deduced from the sole inability of investors to generate excess return over the market's return. Arbitrage strategies are led by rational investors often referred to as 'arbitrageurs' and are possible as long as some assets are mispriced on the market. Though the strict definition of arbitrage refers to some riskless profit opportunities, these arbitrages are not necessarily riskless opportunities, as rational investors on inefficient markets still lack some information.

(3.2.1) No 'free lunch' : if $(\mathbf{x}, \mathbf{y}) \in T$ and $\mathbf{x} = 0$, then $\mathbf{y} = 0$

(3.2.2) No 'free lunch' : $P(\mathbf{0}) = \mathbf{0}$

The traditional axiom of 'no free lunch' can be expressed as in (3.2.1), for any non-negative vector $\mathbf{x}$ of input and any vector $\mathbf{y}$ of output and production possibility set T such that $T = \{(\mathbf{x}, \mathbf{y}) : \mathbf{x}\ can\ produce\ \mathbf{y}\}$. In an approach that assimilates risk to an input, the axiom of 'no free lunch' implies that if no second-order risk characterizes the asset, no return can be generated, which contradicts the existence of assets that are considered free of risk and at the same time generate positive returns (such as T-bills). The axiom of 'no free lunch' therefore precludes the introduction of risk-free assets in the portfolio or the proper analysis of any portfolio with a guaranteed minimum return under an approach that would consider risk as an input. Such drawback could however be overcome by using specific measures of return: in case the excess return over the risk-free rate is used instead of the mean return, this axiom allows taking a risk-free asset into consideration and implies that if no second-order risk characterizes the asset, no excess return can be generated over the risk-free rate. If however second-order risk is defined as an output, the axiom can then be defined as in (3.2.2) and simply implies that for a distribution of returns to be generated, there must be some strictly positive initial investment. However, if short selling or any kind of leverage was allowed it would then be necessary to define an input specific to that kind of investment in order to account for the initial operations implied (borrowing the shares, finding a counterpart, arbitraging). Then, though no single cost can theoretically be incurred, the action undertaken by the investor would be taken into account and the principle of 'no free lunch' would still hold. A zero input vector would remain specific to the very specific case of 'doing nothing' that excludes short selling.

The axiom of the possibility of inaction – sometimes referred to as "doing nothing is feasible" – could be expressed in two ways depending on the assumption of disposability on inputs. On the one hand, the 'raw' axiom of inaction only assumes the possibility of producing no output from a zero vector of input. On the other hand, the 'extended' axiom of inaction (or axiom of 'near' inaction) adds a disposability component to the input variables and assumes the possibility of producing nothing from any non-negative level of input. In an approach that assimilates risk to an input, 'raw' inaction implies that riskless holdings that generate no return belong to the technology set, such that $(\mathbf{0}, \mathbf{0}) \in T$. The origin of the production possibility set can in this case represent any holding generating neither risk nor return (cash holdings or cash equivalent, or any theoretical DMU obtained from the free disposal of a riskless investment). The inclusion of such assets in the technology set can however be hampered by some returns to scale assumptions when no such holding is observed and included in the set directly from the sample set of DMUs. For a classical DEA under VRS or NDRS, the origin does not belong to the set T, except if the axiom of convexity always accepted under the DEA-production approach was rejected.

When risk is assimilated to an output, the representation of cash holdings is the origin of each output set of the output correspondence. The raw axiom of inaction ($\mathbf{0} \in P(\mathbf{x})$ for $\mathbf{x} = \mathbf{0}$) then implies that making no initial investment in any portfolio is possible and will result in no generation of a distribution of returns (and consequently a zero-risk and a zero-return). Still, it does not ensure that the origin of the set belongs to any output set regardless of the level of input (initial investment). It therefore fails to ensure that holding cash or cash equivalents is allowed for any initial amount to be invested. While this axiom relates to the possibility of holding cash under the traditional approach, it only relates to the possibility of doing nothing under the output-oriented approach, which matches its initial meaning.

3.3. Disposability assumptions on outputs

To the best of our knowledge, every portfolio analysis with DEA until now has assumed free disposability on input and output variables (which means on both risks and returns). Free disposability on inputs seems consistent when the initial investment is considered as an input. Free disposability of return when considered as an output is consistent as well, as any return on an investment can be disposed of: once perceived, returns can be kept, reinvested or even wasted. By contrast, the intangible nature of risk seems inconsistent with free disposal. Moreover, when risk is identified as an input variable, assuming free disposability implies the possibility of increasing the level of risk of an investment at a constant level of return. In such case, the addition of any risky asset with a zero mean return to the portfolio would correspond to such increase. But still, such a possibility would depend on the selected risk measure, which prevents us from considering the assumption of free disposability on the risk variable as a generally accepted assumption. Assuming free disposability of risk implies the possibility of full reduction of the risk measure at a constant level of return, and no more the feasibility of unlimited increases in risk. Such reduction will always come at the cost of hedging; costly disposability and an assumption of jointness with the return variables may then be more accurate. Moreover, assuming free disposability on the risk dimension would also prevent any inclusion of risk-loving behaviors in the study as it precludes any projection to 'the right' of the technology set. But as mentioned in Färe & Grosskopf (2003), the disposability assumptions are properties of the technology while the choice of a direction relates to the following step of performance measurement. Only arguments that relate to the definition of the technology should be used to determine which disposability assumptions are appropriate.

The rationale to treat risk as an input is then similar to what makes some authors treat any detrimental variable as an input. The idea that it incurs a cost, together with the natural assumption that decision-makers try to decrease their costs, leads to consider every variable that is to be decreased as an input. In production theory, the same rationale is used in models that assimilate byproducts to freely disposable inputs (introduced by Hailu & Veeman, 2001) with negative shadow price associated to these "bad" outputs. It however implies that no positive value can be attributed to these byproducts, which is a clear limit for portfolio analysis when we consider the progress of the literature on risk-loving behaviors. Moreover, as emphasized in Färe & Grosskopf (2003) considering byproducts as inputs would lead to inconsistencies with both the traditional set of axioms and physical laws. As these byproducts are technically produced by the inputs, they should be considered as outputs. This argument of technical feasibility can also be put forward to support our choice of treating risk measures as output variables, as we considered an initial investment generates a distribution of returns and that both the mean return, the risk or higher moments characterizing this distribution are just statistics of the distribution of a single random variable.

A second limit of these models is that they do not take into consideration any jointness between the so-called 'good' and 'bad' outputs. Including the notion of jointness in the model implies considering the 'bad' variable as an output variable and no more an input variable in order to comply with the definition of jointness and even null jointness, and that's what Färe & Grosskopf (2004) proposed. For this reason, we choose in our approach not to refer to risk or any other output variable as a 'good' or 'bad' outputs but rather identify them as "intended outputs" and "joint outputs". Though we agree on the positivity of shadow prices associated to intended outputs, we leave the characterization of joint outputs as 'good' or 'bad' to the choice of decision-makers according to their own preferences. We then impose no a priori assumption of negativity on shadow prices associated to joint outputs that could potentially be positively valued by some decision-makers. Joint outputs can then either be desired or rejected, but if no uniform preference is assumed among investors it can be interesting to consider the possibility of the expansion path that would increase both. As shown in the following illustrations, even though costly (weak) disposability is not imposed on the undesirable output but free disposability is kept as it would have been if it has been treated as an input, differences in scores and in rankings can be observed due to the application of jointness (when the DMUs are evaluated relative to the risk-averse frontier).

3.4. Introducing jointness to the models

In order to illustrate the changes incurred by the proposed adjustments, we use the data of Morey & Morey (1999). [13] While the original study is multi-horizon, we choose to restrict our illustration to the case of the 3-year horizon only, [14] which allows comparison with the results obtained by Briec, Kerstens & Lesourd (2004). With the traditionally assumed free disposability on risk and return, the portfolio possibility set can be defined as an output set P_1 in (3.4.1) and as P_2 in (3.4.3) after jointness is introduced to the definition of the technology. The related models in (3.4.2) and (3.4.4) are both oriented towards risk reduction only.

(3.4.1) $P_1(\mathbf{x}) = \{(\mu_i, \sigma_{j_i}^2) \ \ \forall i \ : \ \mathbf{q}^{\mathbf{T}}\boldsymbol{\mu} \geqq \mu_i \ , \ \mathbf{q}^{\mathbf{T}}\boldsymbol{\Omega}\mathbf{q} \leqq \sigma_i^2 \ , \ \mathbf{q} \in \Im\}$ for all **x**, with:

- $\boldsymbol{\mu}$ the $(n \times 1)$ vector of mean returns of the n DMUs
- $\boldsymbol{\Omega}$ the $(n \times n)$ matrix of covariances of the n DMUs and $\boldsymbol{\sigma}^{\mathbf{2}}$ the $(n \times 1)$ vector of variance of returns of the n DMUs
- $\mathbf{q}^{\mathbf{T}}$ the transpose of $\mathbf{q}$

(3.4.2) $\min\{\delta^1\}$

s.t.

- $\sum_{j=1}^{n} q_j \mu_j \geq \mu_{j_0}$
- $\sum_{i=1}^{n} \sum_{j=1}^{n} q_i q_j \sigma_{ij} \leq \delta^1 \sigma_{j_0}^2$
- $\sum_{j=1}^{J} q_j = 1$
- $q_j \geq 0$ for all j

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[13] Though the DMUs in this data set are already constituted portfolios (mutual funds) for which we would recommend using an FDH estimator rather than DEA, we keep it here for the sake of the illustration: this data set can easily be found and processed by the reader, and also presents the advantage of having been used for three models orientations: risk reduction and return augmentation in the original paper, and simultaneous risk reduction and return augmentation in Briec, Kertens & Lesourd (2004) and more recently in Briec & Kerstens (2009) to illustrate the changes in the original methodology.

[14] The 3-year seems more consistent than the other time horizons regarding the R^2 of the regression to test the impact of using the expansion path (see note 10).

(3.4.2) $P_2(\mathbf{x}) = \{(\mu_i, \sigma_{j_i}^2)\ \ \forall i:\ \sigma_y(\mathbf{q}^{\mathrm{T}}\boldsymbol{\mu}) \geqq \mu_i\ , \sigma_y \mathbf{q}^{\mathrm{T}}\boldsymbol{\Omega}\mathbf{q} \leqq \sigma_i^2\ , 0 < \sigma_y \leq 1\ , \mathbf{q} \in \Im\}\ \forall \mathbf{x}$

(3.4.4) $\min\{\delta^2\}$

s.t.

- $\sigma_y \sum_{j=1}^{n} q_j \mu_j \geq \mu_{j_0}$
- $\sigma_y \sum_{i=1}^{n} \sum_{j=1}^{n} q_i q_j \sigma_{ij} \leq \delta^2 \sigma_{j_0}^2$
- $\sum_{j=1}^{J} q_j = 1$
- $q_j \geq 0$ for all j
- $0 < \sigma_y \leq 1$

As the introduction of jointness introduces the possibility of reducing risk through the introduction of a riskless asset, portfolios with low return could have carried an even lower risk than the one they currently offer limiting their share of investment in riskless assets. It results in a decrease in efficiency scores for DMUs with the lowest return as well as losses in their respective rank. The efficiency score of the funds IDS Strategy Aggressive B or Seligman for instance decrease by about 35%, and 8 out of 26 funds face a decrease in efficiency scores of more than 20% and are downgraded by up to 8 ranks (all variations in efficiency scores and ranks can be found in Appendix 1). When the model is oriented towards risk reduction and jointness is ignored, the potential risk reduction is underestimated as performance is measured relative to a "weakly efficient" part of the frontier, or what Färe, Grosskopf & Lovell (1994) define as the Weak Efficiency Subset (see Figure 3.5). It implies that the set of optimal portfolios underestimates potential return augmentation, which is partly solved by Briec & al. (2004) who use the EIP function to measure performance relative to the Efficiency Subset (as defined by Färe & al., 1994), sometimes referred to as the "strongly efficient" part of the frontier. But in spite of this result, specific to the data they considered, the use of the EIP function can in no case ensure reaching the Efficiency Subset. On the contrary, as long as costly disposability with jointness (and null jointness) are assumed, performance can be measured relative to the "strongly efficient" part of the frontier whatever the model orientation; it ensures that for any decrease in return risk is reduced as well as much as possible.

On the other hand, this introduction attributes a higher rank to the DMUs with higher levels of risk and returns (the 'risky' funds Keystone Small Co. Growth and PIMCo Adv. Opportunity C for instance move up by 4 ranks each), which provides an additional justification for the use of the expansion path. Similar effects would be observed with a model allowing for limited investment in cash that could be accurate in cases where the definition of the technology implies common restrictions to the share invested in cash. Porter & Gaumnitz (1972) for instance add a maximum requirement of 5% of the portfolio invested in each asset. [15] In this latter case, it is necessary to constraint activity vectors to take be lower than this upper limit. An adjusted axiom of a 'restricted possibility of inaction' can also be proposed to allow for the study of any kind of risky portfolios, from fully riskless to fully risky portfolios. The definition of this 'restricted

[15] The limit of 5% corresponds to the "*normal restriction placed on open-end investment companies and might be considered normal for most other institutional investors*" (Porter & Gaumnitz, 1972).

possibility of inaction' would depend on a factor q_{Cash} to range from a fully restricted to a full access to cash holdings. If $q_{Cash} \in [0,1]$ is defined as the maximum proportion of riskless asset allowed in the portfolio, a restricted possibility of inaction can be defined as in (3.4.5). The particular case $q_{Cash} = 1$ of would then ensure the inclusion of the origin of the output set in the possibility set. A lower limit to the factor of jointness such that $(1 - q_{Cash}) < \sigma_y \leq 1$ would then have similar effects.

(3.4.5) restricted possibility of inaction : $\{\mathbf{0}, \mathbf{0}\} \in P(\mathbf{x})$ and $q_{Cash} \in [0,1]$

3.5. Introducing costly disposability in the models

Costly disposability as it is usually modeled (see Färe & Grosskopf, 2003) consists in three assumptions: jointness, null jointness and costly disposability. We already showed how to introduce jointness in the model (through an 'abatement factor' usually equal to the factor of jointness σ_y on the joint output variables) and proposed to range from limited to null jointness to account for a restricted possibility of inaction. Costly disposability itself can be modeled by an equality in the risk-related constraint as it cannot be freely disposed of. The portfolio possibility set can then be defined as an output set P_3 in (3.5.1). There is no impact on the efficiency scores and of the ranking of DMUs when costly disposability is added to jointness and null jointness for an orientation of risk reduction only. However, for any model oriented towards the expansion path as in (3.5.2), the impact is predictably quite strong: substantial increases in efficiency scores will be observed for DMUs with the highest levels of risk or return (see Appendix 2). All funds but two (20th Century Ultra Investors and Winthrop Focus Aggressive Growth, that already belong to the efficient frontier) improve their efficiency scores, and more than half of the funds (15 funds out of 26) see their efficiency scores improved by more than 20% (8 of them even experience a rise in efficiency score higher than 50%).

(3.5.1) $P_3(\mathbf{x}) = \{(\mu_i, \sigma_{j_i}^2) \;\; \forall i \; : \; \sigma_y(\mathbf{q}^{\mathrm{T}}\boldsymbol{\mu}) \geqq \mu_i \, , \; \sigma_y \mathbf{q}^{\mathrm{T}} \boldsymbol{\Omega} \mathbf{q} = \sigma_i^2 \;\; , 0 < \sigma_y \leq 1 \, , \mathbf{q} \in \Im\} \; \forall \mathbf{x}$

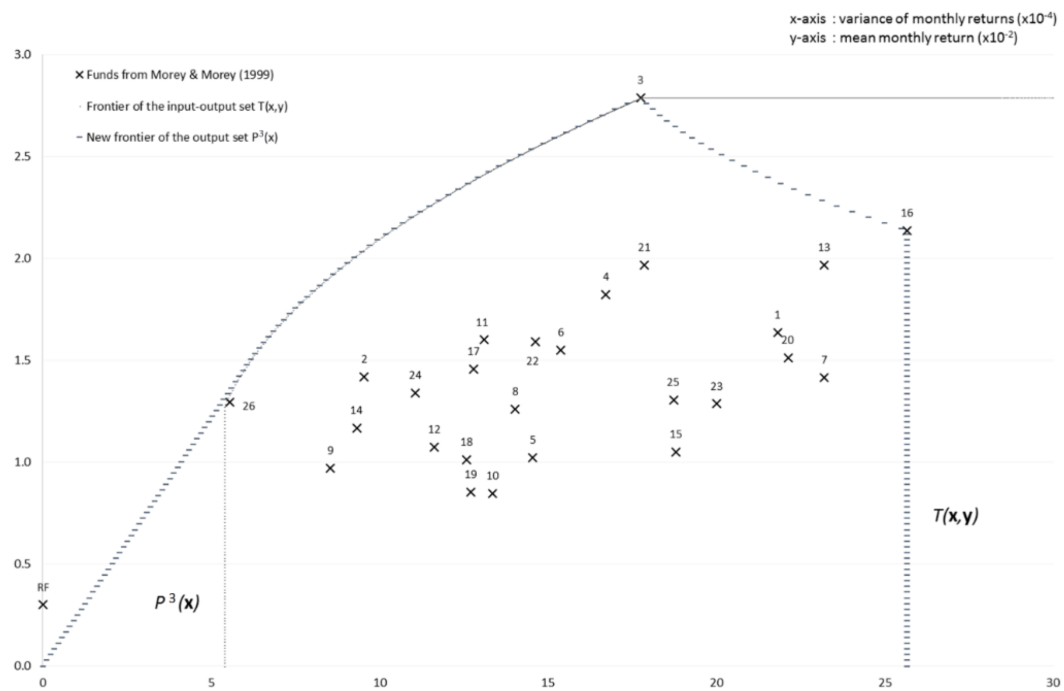
(3.5.2) $\max\{\varphi^3\}$ s.t.

- $\sigma_y \sum_{j=1}^{n} q_j \mu_j \geq \varphi^3 \mu_{j_0}$
- $\sigma_y \left(\sum_{i=1}^{n} \sum_{j=1}^{n} q_i q_j \sigma_{ij}\right) = \varphi^3 \sigma_{j_0}^2$
- $\sum_{j=1}^{J} q_j = 1$
- $q_j \geq 0$ for all j
- $0 < \sigma_y \leq 1$

If the orientation towards an increase in return only was chosen, the above changes would only impact the efficiency and ranking of one DMU in this sample (PIMCo Adv Opportunity C) that carries the highest level of risk but provide the highest attainable return for that level of risk. It would have been deemed inefficient under the traditional approach due to both the assumption of free disposability on return and the model orientation.

Appendix 3 proposed a comparison can also be made between the scores and rankings obtained by the traditional approach based on (3.4.1) and (3.4.2) and the adjusted one proposed in (3.5.1) and (3.5.2). The increase in efficiency is of course very high for the riskier assets regardless of their returns (with an average gain in efficiency score of 69% on the sample and decreases in efficiency scores for only 4 funds out of 26). The observation of such an increase in efficiency scores for the whole sample with the direction of the expansion path may question the consistency of other directions if it is assumed that markets are characterized by efficiency.

Figure 3.5 Portfolio possibility sets under various approaches



3.6. Returns to scale assumptions

Jointness is a key assumption in the context of portfolio analysis: on the one hand, we know that on the financial markets risk can only be reduced through diversification or at the cost of hedging, but once the higher degree of diversification has been reached only the inclusion of some riskless assets like cash in portfolios can further reduce risk. On the other hand, the expected return-risk relationship is positive and can justifies the assumption of jointness as well on an expected return-risk framework. Assuming jointness or imposing the inclusion of cash in the set of DMUs is then relevant for portfolio performance measurement (both solutions will deem similar efficiency measurement in a mean-variance analysis). To this regards, Liu, Zhou, Liu & Xiao (2015) propose to include cash in the analysis by replacing the constraint $\sum_{j=1}^{m} \lambda_j = 1$ (the convexity constraint of activity vectors λ_j for a set of m DMUs j) by a constraint $\sum_{j=1}^{m} \lambda_j \leq 1$ that is actually a mix of the convexity constraint and the Non-Increasing Returns to Scale (NIRS) constraint on some scale parameter θ that should be such that $0 \leq \theta \leq 1$. The new 'convexity constraint' they propose should consequently be written $\sum_{j=1}^{m} z_j \leq 1$ with $z_j = \theta\lambda_j$. Their answer to the need of including a risk-free asset in the portfolio possibility set is to assume NIRS, which includes the origin of the set to the set of possible portfolios but ensures the inclusion of a risk-free asset only if and only if excess returns (above the risk-free rate) are considered instead of returns, which is not the case in most studies and should not be a condition for the set to be consistent. An alternative way of dealing with this matter is to propose jointness as we did in this article.

Moreover, as underlined by Brandouy, Kerstens & Van de Woestyne (2013), "*the very notion of returns to scale may not necessarily be directly transposed to the finance context*". It translates into different implications than in traditional studies with DEA: once redefined the 'financial production process' of generation of a distribution of return by an initial investment, the question rather becomes the following: "to which extent does an increase in the initial investment (input) result in an increase in return and risks (outputs)?". In this case, the notion of returns to scale can very well be transposed to the finance context and is a consistent object of study. But no such link can be made between risk and return and making no returns to scale assumption (assume Variable Returns to Scale – VRS) on output-oriented risk-return models seems more accurate. In a recent contribution, Lamb & Tee (2012) assert that a model with NIRS is appropriate for the study of investment funds. However, this choice is based on the rejection of the other possible RTS assumptions: Constant Returns to Scale (and we suppose Non-Decreasing Returns to Scale) are rejected due to the infeasibility of funds with infinitely high levels of risk and return that are non-attainable in practice, which is actually one of the above arguments to rejected free disposability on risk. The rejection of VRS is due to the fact that it violates the axiom of the possibility of inaction in the input-output space; this problem is solved as soon as risk is defined as an output.

It is also important to notice that the measures chosen to account for input and output variables may once again require a specific treatment: if the distribution of returns is expressed not in monetary units but as rates of return on investment, then scale invariance may even be assumed. Return being traditionally expressed as rates of return, an increase in the quantity invested (free of charges) should remain constant. An assumption of scale invariance can be made on return if expressed as a rate of return and if all additional costs are excluded. In the particular case of some measures like standard deviation of rates of returns, the feasible set of outputs combinations would be the same for all any level of input $\mathbf{x}$. Scale invariance could then be assumed and would translate as $P(\lambda\mathbf{x}) = P(\mathbf{x}) = P$ for all $\lambda > 0$ and $\mathbf{x} \geq \mathbf{0}$.

4. Conclusion

As financial assets differ from traditional DMUs studied in production theory or operations research, performance of such assets cannot consequently be assessed by applying a strictly identical methodology, even if the non-parametric tool (DEA) is the same. From the definition of the underlying technology to the choice of the appropriate model orientation, the whole methodology that has been developed for the use of DEA and almost strictly transposed to the analysis of portfolios is questioned and adjustments are proposed. Prior to the analysis, identifying the purpose of performance measurement is a key concern in a field in which most studies have applied an identical methodology and transposed models from production theory to financial assets of various natures, from individual stocks to portfolios like mutual funds, hedge funds or CTAs. This article emphasizes on the differences between the various purposes of performance measurement and proposes a reflection on how they can impact the traditional set of regularity conditions: it questions for instance the accuracy of assuming when studying already constituted portfolios. The ‘financial production process’ is also redefined as the generation of a distribution of returns over time by an initial investment, which implies to run the analysis the output correspondence for multi-moment frameworks. Beyond its theoretical basis, the assimilation of risk to an output results in convenient consequences on the consistency of the set of axioms and opens the door to new model orientations. By using simple illustrations we show how much the resulting definition of the technology and the new model orientation can impact efficiency scores and rankings of the portfolios. Huge potential increases in efficiency scores question the traditional choice of systematic risk reduction on markets that are theoretically recognized as efficient. Unless the theoretical frameworks are ill-adapted, such variations in efficiency scores lead to reconsider either the assumption of market efficiency, the definition of the technology or the model orientations. The definition of the technology we provide here also allows including a range of preferences that remain ignored by the practitioners in finance, though studied in the literature. Performance can now be measured relative to the frontier of the mean-variance correspondence of the technology set that allows for new model orientations.

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Appendix 1

Impact of the introduction of jointness in the model with a direction of variance reduction on efficiency scores and ranking

Fund	Efficiency score $1-\delta^1$	Rank	Efficiency score $1-\delta^2$	Rank	Variation in efficiency score	Variation in ranking
20th Century Ultra Investors	0.312	21	0.312	18	0%	3
44 Wall Street Equity	0.612	4	0.611	3	0%	1
AIM Aggressive Growth	1.000	1	1.000	1	0%	0
AIM Constellation	0.474	9	0.474	8	0%	1
Alliance Quasar A	0.370	20	0.289	19	-22%	1
Delaware Trend A	0.415	15	0.415	12	0%	3
Evergreen Aggressive Grth A	0.251	26	0.251	25	0%	1
Founders Special	0.384	19	0.369	16	-4%	3
Fund Manager Aggressive Grth	0.630	3	0.466	10	-26%	-7
IDS Strategy Aggressive B	0.403	16	0.260	24	-35%	-8
Invesco Dynamics	0.505	6	0.505	5	0%	1
Keystone Amer Omega A	0.463	11	0.380	15	-18%	-4
Keystone Small Co Grth (S-4)	0.385	18	0.385	14	0%	4
Oppenheimer Target A	0.578	5	0.514	4	-11%	1
Paci®c Horizon Aggr Growth	0.286	23	0.229	26	-20%	-3
PIMCo Adv Opportunity C	0.403	17	0.403	13	0%	4
Putnam Voyager A	0.468	10	0.468	9	0%	1
Security Ultra A	0.428	13	0.331	17	-23%	-4
Seligman Capital A	0.424	14	0.275	22	-35%	-8
Smith Barney Aggr Growth A	0.281	24	0.281	21	0%	3
State St. Research Capital C	0.501	8	0.501	6	0%	2
SteinRoe Capital Opport	0.450	12	0.450	11	0%	1
USAA Aggressive Growth	0.270	25	0.264	23	-2%	2
Value Line Leveraged Gr Inv	0.502	7	0.496	7	-1%	0
Value Line Spec Situations	0.291	22	0.286	20	-2%	2
Winthrop Focus Aggr Growth	0.976	2	0.955	2	-2%	0

Appendix 2

Impact of the introduction of jointness and costly disposability in the output-oriented model on efficiency scores and ranking

Fund	Efficiency score $1/\varphi^1$	Rank	Efficiency score $1/\varphi^3$	Rank	Variation in efficiency score	Variation in ranking
20th Century Ultra Investors	0.587	7	0.850	6	45%	1
44 Wall Street Equity	0.509	13	0.523	20	3%	-7
AIM Aggressive Growth	1.000	1	1.000	1	0%	0
AIM Constellation	0.654	6	0.767	9	17%	-3
Alliance Quasar A	0.367	22	0.567	18	54%	4
Delaware Trend A	0.556	10	0.673	13	21%	-3
Evergreen Aggressive Grth A	0.508	14	0.904	4	78%	10
Founders Special	0.453	18	0.573	17	27%	1
Fund Manager Aggressive Grth	0.348	24	0.401	26	15%	-2
IDS Strategy Aggressive B	0.304	26	0.520	21	71%	5
Invesco Dynamics	0.574	8	0.643	15	12%	-7
Keystone Amer Omega A	0.386	20	0.483	24	25%	-4
Keystone Small Co Grth (S-4)	0.706	4	0.914	3	30%	1
Oppenheimer Target A	0.419	19	0.464	25	11%	-6
Paci®c Horizon Aggr Growth	0.377	21	0.733	10	95%	11
PIMCo Adv Opportunity C	0.767	2	1.000	1	30%	1
Putnam Voyager A	0.523	12	0.603	16	15%	-4
Security Ultra A	0.364	23	0.490	23	35%	0
Seligman Capital A	0.306	25	0.495	22	62%	3
Smith Barney Aggr Growth A	0.543	11	0.862	5	59%	6
State St. Research Capital C	0.706	4	0.824	7	17%	-3
SteinRoe Capital Opport	0.571	9	0.670	14	17%	-5
USAA Aggressive Growth	0.462	17	0.780	8	69%	9
Value Line Leveraged Gr Inv	0.480	15	0.540	19	12%	-4
Value Line Spec Situations	0.468	16	0.730	11	56%	5
Winthrop Focus Aggr Growth	0.726	3	0.726	12	0%	-9

Appendix 3

Impact of the introduction of jointness and costly disposability in the model and the change in model orientation on efficiency scores and ranking

Fund	Variation in efficiency score from a direction of risk minimization to the expansion path	Variation in ranking
20th Century Ultra Investors	172%	15
44 Wall Street Equity	-15%	-16
AIM Aggressive Growth	0%	0
AIM Constellation	62%	0
Alliance Quasar A	53%	2
Delaware Trend A	62%	2
Evergreen Aggressive Grth A	260%	22
Founders Special	49%	2
Fund Manager Aggressive Grth	-36%	-23
IDS Strategy Aggressive B	29%	- 5
Invesco Dynamics	27%	- 9
Keystone Amer Omega A	4%	-13
Keystone Small Co Grth (S-4)	137%	15
Oppenheimer Target A	-20%	-20
Paci®c Horizon Aggr Growth	156%	13
PIMCo Adv Opportunity C	148%	16
Putnam Voyager A	29%	- 6
Security Ultra A	15%	-10
Seligman Capital A	17%	- 8
Smith Barney Aggr Growth A	207%	19
State St. Research Capital C	65%	1
SteinRoe Capital Opport	49%	- 2
USAA Aggressive Growth	188%	17
Value Line Leveraged Gr Inv	8%	-12
Value Line Spec Situations	150%	11
Winthrop Focus Aggr Growth	-26%	-10

Average variation	69%