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WEAKLY CONTINUOUS SECURITY IN DISCONTINUOUS AND NONQUASICONCAVE GAMES: EXISTENCE AND CHARACTERIZATION*

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Abstract

This paper investigates the existence of pure strategy Nash equilibria in discontinuous and nonquasiconcave games. We introduce a new notion of continuity, called *weakly continuous security*, which is weaker than the most known weak notions of continuity, including the continuous security of Barelli and Meneghel [2013], C-secure of McLennan, Monteiro and Tourky [2011], generalized weakly transfer continuity of Nessah [2011], generalized better-reply security of Carmona [2011], Barelli and Soza [2009], Barelli and Meneghel [2013], lower single deviation property of Reny [2009], better-reply security of Reny [1999] and the results of Prokopovych [2011, 2012] and Carmona [2009]. We show that a compact, convex and weakly continuous secure Hausdorff locally convex topological vector space game has a pure strategy Nash equilibrium. Besides, it holds in a large class of discontinuous games.

Keywords: Discontinuous games, nonquasiconcave games, pure strategy Nash equilibrium, weakly continuous security, pseudo upper semicontinuity, generalized payoff security, weakly reciprocal upper semicontinuity.

JEL Classifications: C72 - Noncooperative Games

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1 INTRODUCTION

This paper presents an existence of pure strategy Nash equilibrium theorem, which characterizes the existence of pure strategy equilibrium in games which the strategy space is not necessary compact and/or convex and payoff functions are may be discontinuous and/or nonquasiconcave. Debreu [1952] then showed that games possess a pure strategy Nash equilibrium if the game is convex, compact, continuous and quasiconcave. Game theory has then been successfully applied in many areas in economics including oligopoly theory, social choice theory, and incentive mechanism design theory. These applications lead researchers from different fields to investigate the possibility of weakening equilibrium existence conditions to further enlarge its domain of applicability.

Nishimura and Friedman [1981] considered the existence of Nash equilibria in games where the payoff functions are not quasi-concave but satisfying a strong condition. Dasgupta and Maskin [1986] established the existence of pure and mixed strategy Nash equilibria in games where the strategy sets are convex and compact, and payoff functions are quasiconcave, upper semicontinuous and graph continuous by using an approximation technique. Simon [1987] and Simon and Zame [1990] used a similar approach to consider the existence of mixed strategy Nash equilibria in discontinuous games. Simon and Zame [1990] showed that if one is willing to modify the vector of payoffs at points of discontinuity so that they correspond to points in the convex hull of limits of nearby payoffs, then one can ensure a mixed strategy equilibrium of such a suitably modified game. Vives [1990] established the existence of Nash equilibria in games where payoffs are upper semicontinuous and satisfy certain monotonicity properties. Baye, Tian and Zhou [1993] provided necessary and sufficient conditions for the existence of pure strategy Nash equilibria and dominant strategy equilibria in noncooperative games which may have discontinuous and/or non-quasiconcave payoffs. It is shown that diagonal transfer quasiconcavity is necessary, and further, under diagonal transfer continuity and compactness, sufficient for the existence of pure strategy Nash equilibrium. Both transfer quasiconcavity and diagonal transfer continuity are very weak notions of quasiconcavity and continuity and use a basic idea of transferring nonequilibrium strategies to a securing profile of strategies. Reny [1999] established the existence of Nash equilibria in compact and quasiconcave games where the game is better-reply secure, which is a weak notion of continuity. Reny [1999] showed that better-reply security can be imposed separately as reciprocal upper semicontinuity introduced by Simon [1987] and payoff security. Bagh and Jofre [2006] further weakened reciprocal upper semicontinuity to weak reciprocal upper semicontinuity and showed that it, together with payoff security, implies better-reply security. Reny [2009] introduced a game property that is weaker than better-reply security, called the lower single-deviation property and proved that if a game is bounded, convex, compact, quasiconcave and has the lower single-deviation property, then it possesses a pure strategy Nash equilibrium. Prokopovych [2011] introduced the transfer reciprocal upper semicontinuity and established the existence of Nash equilibrium in compact and quasiconcave games where the game is payoff secure and transfer reciprocal upper semicontinuous. Carmona [2011] introduced the weak better-reply security. He showed that a bounded, convex, compact, quasiconcave game and weakly better-reply secure has a Nash equilibrium. He also proved that, when players' action spaces are metric and locally convex, this implies the existence results of Reny [1999] and Carmona [2009] and it is equivalent to the result of Barelli and Soza [2009]. Nessah [2011] introduced the generalized weak transfer continuity and showed that a bounded, compact, convex, quasiconcave and generalized weak transfer continuous game has an equilibrium.

Tian [2009] characterized the existence of equilibria in games with general strategy spaces and payoffs. He established a single condition, called recursive diagonal transfer continuity, which is both necessary and sufficient for the existence of equilibria in games with arbitrary compact strategy spaces and payoffs. McLennan, Monteiro and Tourky [2011] characterized the existence of Nash equilibrium in compact and convex games and established a single condition, called MR-secure, that is both necessary and sufficient for the existence of equilibrium in games under the compactness and convexity conditions. More recently, Barelli and Meneghel [2013] introduced the continuous security condition and proved that a convex, compact and continuously secure game has a pure-strategy Nash equilibrium. Other papers on existence of equilibrium with discontinuous payoff functions include Baye, Kovenock and de Vries [2012], Carmona [2012], Carmona and Podczeck [2012], Prokopovych [2012], Balder [2011], de Castro [2011], Reny [2011], Carbonell-Nicolau [2011], Carmona and Podczeck [2009], Bich [2009], Duggan [2007], Monteiro and Page [2007, 2008], Jackson and Swinkels [2005] and Athey [2001]. Like other existing characterization results, this is mainly for the purpose of providing a way of understanding equilibrium and identifying whether or not a game has an equilibrium. In general, the weaker a condition in an existence theorem, the harder it is to verify whether the conditions are satisfied in a particular game.

This paper investigates the existence of pure strategy Nash equilibria in discontinuous and/or nonquasiconcave games. We introduce a new notion of very weak continuity, called *weakly continuous secure*, which holds in a large class of discontinuous games. Roughly speaking, a game is weakly continuous secure if for every nonequilibrium strategy $\overline{x}$, there is a neighborhood $\mathcal{V}$ of $\overline{x}$ and a well-behaved correspondence $\phi_{\overline{x}} : \mathcal{V} \twoheadrightarrow X$ so that for every strategy deviation profile z nonequilibrium in $\mathcal{V}$, the strategy z_j is not in the convex hull of strategy (t_j, z) which dominates all strategies (y_j, x) in the graph of $\phi_{\overline{x},j}$, for some player j. We establish that the game has a pure strategy Nash equilibrium if and only if there exists a compact, convex and dominant subset X^0 in X such that G is weakly continuous secure on X^0 under the boundedness of $G = (X_i, u_i)_{i \in I}$.

The condition of g-weak continuous security if more flexible and easily to check compared to the recursively diagonal transfer continuity of Tian [2009], continuous security of Barelli and Meneghel [2013] and the multiply restrictionally security of McLennan, Monteiro and Tourky

[2011]. We show that it strictly generalizes Barelli and Meneghel [2013], McLennan, Monteiro and Tourky [2011], Bich and Laraki [2012], Carmona [2011, 2009], Barelli and Soza [2009], Reny [1999, 2009], Nessah [2011], and Prokopovych [2011, 2012]. In Reny [1999] it is shown that better-reply security can be imposed separately as two conditions. We introduce here two new conditions called g-pseudo upper semicontinuity and g-generalized payoff security. We prove that the g-pseudo upper semicontinuity together with g-generalized payoff security, implies g-weakly continuous security under the quasiconcavity of payoffs, and we show that 0-pseudo upper semicontinuity is weaker than the weakly reciprocal upper semicontinuity of Bagh and Jofre [2006] and transfer reciprocal upper semicontinuity of Prokopovych [2011]. Consequently, we strictly generalize the results of Proposition 1 of Bagh and Jofre [2006], Corollaries 3.3-3.4 of Reny [1999] and Corollary 8.5 of McLennan, Monteiro and Tourky [2011]. These conditions are satisfied in many economic games and are often simple to check.

The remainder of the paper is organized as follows. Section 2 describes the notations. Subsection 3.1 introduces the notion of weakly continuous security, provides the main result on the characterization and existence of pure strategy Nash equilibrium, examples illustrating the theorems are also given. Subsection 3.2 offers the sufficient conditions for the weak continuous security. Section 4 describes the related results. Section 5 presents some application examples. Section 6 will briefly present the conclusion.

2 Preliminaries and definitions

Consider the following noncooperative game in a normal form: $G = (X_i,\ u_i)_{i\in I}$ where $I = \{1, ..., n\}$ is a finite set of players, X_i is player i's strategy space that is a nonempty subset of a Hausdorff locally convex topological vector space E_i, and u_i is player i's payoff function from the set of strategy profiles $X = \prod\limits_{i\in I} X_i$ to $\mathbb{R}$. For each player $i \in I$, denote by $-i$ all players rather than player i. Also denote by $X_{-i} = \prod\limits_{j\neq i} X_j$ the set of strategies of the players in $-i$. The graph of the game is $\Gamma = \{(x, u) \in X \times \mathbb{R}^n : u_i(x) = u_i,\ \forall i \in I\}$. The closure of Γ in $X \times \mathbb{R}^n$ is denoted by cl(Γ). The frontier of Γ, which is the set of points that are in cl(Γ) but not in Γ, is denoted by Fr(Γ). We say that a game $G = (X_i,\ u_i)_{i\in I}$ is compact, convex and bounded, respectively if, for all $i \in I$, X_i is compact and convex, and u_i is bounded on X. We say that a game $G = (X_i,\ u_i)_{i\in I}$ is quasiconcave if, for every $i \in I$, X_i is convex and the function u_i is quasiconcave in x_i.

We say that a strategy profile $\overline{x} \in X$ is a *pure strategy Nash equilibrium* of a game G if, for each $i \in I$, and $y_i \in X_i$, we have $u_i(y_i, \overline{x}_{-i}) \leq u_i(\overline{x})$.

A correspondence $C : Z \twoheadrightarrow E$ is said to be upper hemicontinuous at the point $z \in Z$ if, for any open set $\mathcal{V}$ of $C(z)$ there exists a neighborhood $\mathcal{V}_z$ of z such that for all $z' \in \mathcal{V}_z$, $C(z') \subset \mathcal{V}$. C is said closed if its graph is closed in $Z \times E$ (Graph$(C) = \{(z, y) \in Z \times E :\ y \in C(z)\}$). When Z is compact Hausdorff, then C is closed if and only if it is upper hemicontinuous and closed valued. Denote also by coA the convex hull of A.

3 MAIN RESULTS

In this section, we examine the existence of a pure strategy Nash equilibrium in discontinuous games. We introduce a new notion of very weak continuity, called *weakly continuous secure*, which holds in a large class of discontinuous games. We characterize the existence of pure strategy Nash equilibrium and we introduce some sufficient conditions.

3.1 WEAKLY CONTINUOUS SECURITY: EXISTENCE AND CHARACTERIZATION

Let us consider a correspondence $C : X \twoheadrightarrow Y$. C is said to be a well-behaved correspondence if it is upper hemicontinuous with nonempty, convex and closed values. For each player $i \in I$, let $\overline{u}_i$ be real-valued function defined on $X_i \times X$ and $\overline{u} = (\overline{u}_1, ..., \overline{u}_n) \in \mathbb{R}^n$. Let X^0 be a nonempty subset of X.

DEFINITION 3.1 A game $G = (X_i, u_i)_{i \in I}$ is said to be $\overline{u}$-*weakly continuous secure* on X^0 if whenever $\overline{x} \in X^0$ is not an equilibrium, there exist a neighborhood $\mathcal{V} \subseteq X^0$ of $\overline{x}$ and a well-behaved correspondence $\phi_{\overline{x}} : \mathcal{V} \twoheadrightarrow X^0$ so that for each $z \in \mathcal{V}$ nonequilibrium, there exists a player j so as for each $(x, y_j) \in \text{Graph}(\phi_{\overline{x},j})$,

$$z_j \notin co\{t_j \in X_j^0 \text{ such that } \overline{u}_j(t_j, z) \geq \overline{u}_j(y_j, x)\}.$$

A game is weakly continuous secure if for every nonequilibrium strategy $\overline{x}$, there is a neighborhood $\mathcal{V}$ of $\overline{x}$ and a well-behaved correspondence $\phi_{\overline{x}} : \mathcal{V} \twoheadrightarrow X$ so that for every strategy deviation profile z nonequilibrium in $\mathcal{V}$, the strategy z_j is not in the convex hull of strategy (t_j, z) which dominates all strategies (y_j, x) in the graph of $\phi_{\overline{x},j}$, for some player j. In Section 4, we show that $\overline{u}$-weakly continuous security is weaker than *continuous security* of Barelli and Meneghel [2013], C-*secure* of McLennan, Monteiro and Tourky [2011], *generalized deviation property* of Bich and Laraki [2012], *generalized better-reply security* of Barelli and Meneghel [2013], Carmona [2011], Barelli and Soza [2009], *lower single deviation property* of Reny [2009], *better-reply security* of Reny [1999] and *generalized weakly transfer continuity* of Nessah [2011].

REMARK 3.1 If the game G is quasiconcave, the condition for each $(x, y_j) \in \text{Graph}(\phi_{\overline{x},j})$, $z_j \notin co\{t_j \in X_j \text{ such that } \overline{u}_j(t_j, z) \geq \overline{u}_j(y_j, x)\}$ becomes $\overline{u}_j(y_j, x) > \overline{u}_j(z_j, z)$, for each $(x, y_j) \in \text{Graph}(\phi_{\overline{x},j})$.

REMARK 3.2 In the definition of continuous security of Barelli and Meneghel [2013], the deviation $z \in \mathcal{V}$ is not imposed to be nonequilibrium contrary to the definition of weak continuous security. Then, if $\mathcal{V}$ contains an equilibrium, this fails the definition of point secure. To illustrate this point, consider the following example.

EXAMPLE 3.1 Consider the two-player game with the following payoff functions defined on $[0,1]\times[0,1]$ by:

$$u_i(x_1,x_2)=\begin{cases}1 & \text{if } x_i>\frac{1}{2}\\ 0 & \text{otherwise.}\end{cases}$$

This game is compact, convex and quasiconcave, but it is not continuous security. Indeed, let $x=(\frac{1}{2},\frac{1}{2})$. Clearly x is not a Nash equilibrium. Therefore, for each neighborhood $\mathcal{V}$ of x and each well-behaved correspondence $\phi_x:\mathcal{V}\twoheadrightarrow X$, there exists $\tilde{z}\in\mathcal{V}$ with $\tilde{z}_1=\tilde{z}_2>\frac{1}{2}$ such that $u_j(\tilde{z})=1\geq u_j(y_j,t_{-j})$, for each $j=1,2$, and for each $(t,y_j)\in\phi_{x,j}$. Note that $\tilde{z}\in\mathcal{V}$ is a Nash equilibrium. However, this game is weakly continuous secure. To prove it, it is sufficient to let the correspondence $\varphi:\mathcal{V}\twoheadrightarrow X$ defined by $\varphi(z)=\{(1,1)\}$ where $\mathcal{V}$ is a neighborhood of x, for each $x\in X$ nonequilibrium and $\overline{u}\equiv u$.

Tian [2009] introduced the notion of *upsetting* defined as follows: the strategy y upsets the strategy x if there is a player i such that $u_i(y_i,x_{-i})>u_i(x)$. Based on the concept of *upsetting*, we introduce the following dominance of sets as follows. The set X^0 dominates X if for each $x\in X^0$ and whenever x is upset by a strategy in $X\backslash X^0$, then it is upset by a strategy in X^0.

The following theorem characterizes the existence of pure strategy Nash equilibrium without convexity and/or compactness of X.

THEOREM 3.1 The $G=(X_i,u_i)_{i\in I}$ has a pure strategy Nash equilibrium if and only if there exists a nonempty, dominant, compact and convex subset X^0 of X such that G is $\overline{u}$-weakly continuous secure on X^0.

REMARK 3.3 The condition of g-weak continuous security if more flexible and easily to check compared to the recursively diagonal transfer continuity of Tian [2009], continuous security of Barelli and Meneghel [2013] and the multiply restrictionally security of McLennan, Monteiro and Tourky [2011].

REMARK 3.4 Theorem 3.1 characterizes the existence of pure strategy equilibrium in games which the strategy space is not necessary compact and/or convex and payoff functions are may be discontinuous and/or nonquasiconcave. Then, it generalizes the most existing results on the existence of pure strategy Nash equilibrium.

The Necessity of Theorem 3.1 is particulary useful to verify the nonexistence of pure strategy Nash equilibrium. Example 3.2 shows that the game is not weakly continuous secure and since it is compact and convex then according to Theorem 3.1, the considered game has no pure strategy Nash equilibrium. Let us consider the following example.

EXAMPLE 3.2 Consider the following two players on the unit square $X_1 = X_2 = [0,1]$ that was considered by Dasgupta and Maskin [1986]. For player $i = 1,2$ and $x = (x_1, x_2) \in X = [0,1]^2$, the payoff functions are:

$$u_i(x) = \begin{cases} 0, & \text{if } x_1 = x_2 = 1 \\ x_i, & \text{otherwise} \end{cases} \quad i = 1,2$$

This game is bounded and quasiconcave. Let us show that it is not $\overline{u}$-weakly continuous secure on X^0, for each X^0, dominant, compact and convex set of X. Let X^0 be a compact, convex and dominant subset of X[1] and $\overline{x} \in X^0$ defined by $\overline{x} = (\max_{x_1 \in X_1^0} x_1, \max_{x_2 \in X_2^0} x_2)$. Let $\overline{u}_i(y_i, x) = \sup_{\mathcal{V}_x \in \Omega(x)} \sup_{\varphi_i \in W_{\mathcal{V}_x}(y_i,x)} \inf_{(t_i,z) \in \text{Graph}(\varphi_i)} u_i(t_i, z_{-i})$ [2] in Definition 3.1 and for any neighborhood $\mathcal{V}$ of $\overline{x}$, any well-behaved correspondence $\varphi : \mathcal{V} \twoheadrightarrow X^0$, let $\beta_i = \inf_{(t,y_i) \in \text{Graph}(\varphi_i)} \overline{u}_i(y_i, t_{-i}) =$

$$\begin{cases} 0, & \text{if } (1,1) \in \text{Graph}(\varphi_i) \\ y_i, & \text{otherwise.} \end{cases}$$

There is a $z \in \mathcal{V} \cap X^0$ with $z_i \geq \overline{x}_i$ for each $i = 1,2$ such that $\overline{u}_j(z) \geq \beta_j$, for each $j = 1,2$. Thus, this game is not $\overline{u}$-weakly continuous secure on X^0. So by Theorem 3.1, this game does not have a pure strategy Nash equilibrium.

PROOF OF THEOREM 3.1. *Necessity*: Suppose that the game G has a Nash equilibrium $\overline{x} \in X$. Let $x \in X$ be nonequilibrium and $X^0 = \{\overline{x}\}$. Since $\overline{x}$ is a Nash equilibrium, then it can not be upset by a strategy in $X \backslash X^0$. Therefore, there is a neighborhood $\mathcal{V}$ of x, a well-behaved correspondence $\phi_x : \mathcal{V} \twoheadrightarrow X$ defined by $\phi_x(z) = \{\overline{x}\}$, such that for each $z \in \mathcal{V}$ nonequilibrium, there is a player j so that $z_j \neq \overline{x}_j$. Consequently $z_j \notin co\{t_j \in X_j^0$ such that $\overline{u}_j^g(t_j, z) \geq \overline{u}_j^g(y_j, x)$ for each $(x, y_j) \in \text{Graph}(\phi_{\overline{x},j})\}$.

Sufficiency: Let X^0 be a nonempty dominant, compact, convex subset of X such that G is weakly continuous secure on X^0. Assume that the game $G^0 = (X_i^0, u_i)_{i \in I}$ has a pure strategy Nash equilibrium $\overline{x} \in X^0$. If $\overline{x}$ is not a Nash equilibrium of G, then there is a player j, a strategy $\tilde{y} \in X$ so that $u_j(\tilde{y}_j, \overline{x}_{-j}) > u_j(\overline{x})$. The strategy $\tilde{y} \notin X^0$ otherwise we have a contradiction with $\overline{x}$ is a Nash equilibrium in X^0. Thus, $\tilde{y} \in X \backslash X^0$ upsets $\overline{x} \in X^0$, and by dominance of X^0, $\overline{x}$ is upset by a strategy in X^0, which is a contradiction with $\overline{x}$ is a Nash equilibrium in X^0. Hence, $\overline{x}$ is a Nash equilibrium of G.

Assume that there is no equilibrium in X^0. Then by weakly continuous security, for each $x \in X$, there is a neighborhood $\mathcal{V}_x \subseteq X^0$ and a well-behaved correspondence $\varphi_x : \mathcal{V}_x \twoheadrightarrow X^0$ so that for each $z \in \mathcal{V}_x$ nonequilibrium, there exists a player j so that for each $(x', y'_j) \in \text{Graph}(\varphi_{x,j})$, we have

$$z_j \notin co\{t_j \in X_j^0 \text{ such that } \overline{u}_j(t_j, z)_j \geq \overline{u}_j(y'_j, x')\}.$$

[1] The compact, convex and dominant subsets of X are of the form $[\alpha, 1] \times [\beta, 1]$, with $\alpha, \beta \in [0,1)$

[2] $\Omega(x)$ is the set of all open neighborhoods of x, for each $x \in X$. For each $(y_i, x) \in X_i \times X$, $W_{\mathcal{V}}(y_i, x)$ is the set of all well-behaved correspondences $\varphi_i : \mathcal{V} \twoheadrightarrow X_i$ that satisfy $(x, y_i) \in \text{Graph}(\varphi_i)$

Thus, we obtain a collection $\{(\mathcal{V}_x, \varphi_x)\}_{x\in X}$ where $\{\mathcal{V}_x\}_{x\in X}$ form an open cover of X. Since X is compact, then it can be extract a finite subcollection $\{(\mathcal{V}_{x^k}, \varphi_{x^k})\}_{k\in K}$ (K is a finite set), so as for each $k \in K$ and for each $z \in \mathcal{V}_{x^k}$ nonequilibrium, there exists a player j so that for each $(x, y_j) \in \text{Graph}(\varphi_{x^k,j})$, we have

$$z_j \notin co\{t_i \in X_i \text{ such that } \overline{u}_j(t_j, z) \geq \overline{u}_j(y_j, x)\}. \tag{3.1}$$

Define k dominates k' for i if for each $(x, y_i) \in \text{Graph}(\varphi_{x^k,i})$, there is $(x', y_i') \in \text{Graph}(\varphi_{x^{k'},i})$ such that

$$\overline{u}_i(y_i, x) \geq \overline{u}_i(y_i', x').$$

This dominance relation is complete, reflexive and transitive. For each $y \in X$, let $\mathcal{V}_y = \left(\bigcap_{k\in K: y\in\mathcal{V}_{x^k}} \mathcal{V}_{x^k}\right) \bigcap \left(\bigcap_{k\in K: y\notin\mathcal{V}_{x^k}} \text{cl}(\mathcal{V}_{x^k})^c\right)$ [3], a neighborhood of y and for each player $i \in I$, let $f_i(y)$ be any member of the set $\{k \in K : y \in \mathcal{V}_{x^k}\}$ that dominate every other member of the set.

For each $y \in X$, let $\mathcal{U}_y = \mathcal{V}_y \bigcap \left(\bigcap_{i\in I} \mathcal{V}_{x^{f_i(y)}}\right)$ be an open neighborhood of y, and $\phi_y : \mathcal{U}_y \twoheadrightarrow X$ defined by

$$\phi_{y,i}(t) = \varphi_{x^{f_i(y)},i}(t) \tag{3.2}$$

Note that ϕ_y is well-behaved correspondence. We have for each player i, for each $(x', y_i') \in \text{Graph}(\phi_y)$ ($x' \in \mathcal{U}_y \subseteq \mathcal{V}_{x^{f_i(y)}}$) and for each $k \in K$ with $y \in \mathcal{V}_{x^k}$, there exists $(x'', y_i'') \in \text{Graph}(\varphi_{x^k,i})$ such that $\overline{u}_i(y_i', x') \geq \overline{u}_i(y_i'', x'')$ (Indeed, $y \in \mathcal{V}_{x^k}$ and $f_i(y)$ dominates k for i). Then, we obtain a collection of couple $\{(\mathcal{U}_y, \phi_y)\}_{y\in X}$ where $\{U_y\}_{y\in X}$ form an open cover of X. Since X is compact, then it can be extract a finite subcollection $\{(\mathcal{U}_{y^h}, \phi_{y^h})\}_{h\in H}$ (H is a finite set) so that for each $h \in H$, for each player i, for each $k \in K$ with $y^h \in \mathcal{V}_{x^k}$, and for each $(x', y_i') \in \text{Graph}(\phi_{y^h})$, there exists $(x'', y_i'') \in \text{Graph}(\varphi_{x^k,i})$ such as

$$\overline{u}_i(y_i', x') \geq \overline{u}_i(y_i'', x''). \tag{3.3}$$

Let $\{\beta_h\}_{h\in H}$ be a partition of the unity subordinate to $\{\mathcal{U}_{y^h}\}_{h\in H}$ and consider the following correspondence $\Psi : X^0 \twoheadrightarrow X^0$ defined by

$$\Psi(z) = \sum_{h\in H} \beta_h(z)\phi_{y^h}(z).$$

Then, it is easy to see that Ψ is an upper hemicontinuous correspondence with nonempty, closed and convex values (by (3.2)). Thus, Ψ has a fixed point $\overline{z} \in X^0$ by the Kakutani-Fan-Glicksberg Theorem (see Aliprantis and Border [2006], Corollary 17.55).

[3] Let $A \subseteq X$, then denote by A^c the complementary set of A in X, i.e. $A^c = \{z \in X,\ x \notin A\}$

Let $J = \{h \in H$ such that $\beta_h(\overline{z}) > 0\}$, then $\overline{z} = \sum_{h \in H} \beta_h(\overline{z})\widetilde{y}^h$ where $\widetilde{y}^h \in \phi_{y^h}(\overline{z})$. Let $h \in J$ and $k \in K$ with $\overline{z} \in \mathcal{V}_{x^k}$, then $\overline{z} \in \mathcal{U}_{y^h} \bigcap \mathcal{V}_{x^k}$. Therefore for each player i, there exists $(x^{i,h}, y_i^{i,h}) \in$ Graph$(\varphi_{x^k,i})$ such as by (3.3), we have

$$\overline{u}_i(\tilde{y}_i^h, \overline{z}) \geq \overline{u}_i(y_i^{i,h}, x^{i,h}). \tag{3.4}$$

Let $h_0 \in J$ so that $\overline{u}_i(y_i^{i,h_0}, x^{i,h_0}) \leq \overline{u}_i(y_i^{i,h}, x^{i,h})$, for each $h \in J$. Thus by (3.4), for each player $i \in I$, we have $\overline{z}_i \in co\{t_i \in X_i$ such that $\overline{u}_i(t_i, \overline{z}) \geq \overline{u}_i(y_i^{i,h_0}, x^{i,h_0})\}$ which is a contradiction with (3.1). ■

When $X^0 = X$, we have the following corollary.

COROLLARY 3.1 *Assume that the game $G = (X_i, u_i)_{i \in I}$ is bounded, compact and convex. If in addition G is $\overline{u}$-weakly continuous secure, then G has a pure strategy Nash equilibrium.*

Let us consider the following examples.

EXAMPLE 3.3 Consider a timing game between two players on the unit square $X_1 = X_2 = [0, 1]$ that was considered by Reny [1999] and Bagh and Jofre [2006]. For player $i = 1, 2$ and $t = (t_1, t_2) \in X = [0, 1]^2$, let the payoff functions for the players be given by

$$u_i(t_1, t_2) = \begin{cases} l_i(t) = 10, & \text{if } t_i < t_{-i} \\ k_i(t), & \text{if } t_i = t_{-i} = t \\ m_i(t_{-i}) = -10, & \text{if } t_i > t_{-i}, \end{cases}$$

where $k_i(t) = 10$ when $t < \frac{1}{2}$ and $k_i(t) = 0$ when $t \geq \frac{1}{2}$. It is clear that this game is bounded, compact and quasiconcave.

This game is not continuous secure[4]. Indeed, consider $t = (\frac{1}{2}, \frac{1}{2}) \in X$. Then, t is not an equilibrium. For any neighborhood $\mathcal{V}$ of t, any correspondence $\varphi : \mathcal{V} \twoheadrightarrow X$, and for each $\alpha \in \mathbb{R}^n$, we have: If for each $j = 1, 2$, $\alpha_j \leq 10$, choose $z \in \mathcal{V}$ with $z_1 = z_2 < \frac{1}{2}$. Therefore $u_i(z) = 10 \geq \alpha_i$, for each $i = 1, 2$. If $\alpha_j > 10$ for some $j = 1, 2$, then $u_j(t_j, z_{-j}) \leq 10 < \alpha_j$, for each $t_j \in \varphi_j(z)$. Thus, this game is not continuous secure, so that Theorem 2.2 and Proposition 2.4 in Barelli and Meneghel [2013] and Proposition 2.7 of McLennan, Monteiro and Tourky [2011] cannot be applied.

[4]A game $G = (X_i, u_i)_{i \in I}$ is said to be *continuous secure* if whenever $\overline{x} \in X$ is not an equilibrium, there exist a neighborhood $\mathcal{V}$ of $\overline{x}$, $\alpha \in \mathbb{R}^n$, and a well-behaved correspondence $\phi_{\overline{x}} : \mathcal{V} \twoheadrightarrow X$ so that

(a) for each $t \in \mathcal{V}$, and $i \in I$, we have $\phi_{\overline{x},i}(t) \subseteq B_i(t, \alpha_i)$, and

(b) for each $z \in \mathcal{V}$, there exists a player j for whom, $z_j \notin coB_j(z, \alpha_j)$.

Where $B_i(x, \alpha_i) = \{y_i \in X_i$ such that $u_i(y_i, x_{-i}) \geq \alpha_i\}$.

This game does not have the generalized deviation property[5]. Indeed, consider $t = (\frac{1}{2}, \frac{1}{2}) \in X$. Then, t is not an equilibrium. Any neighborhood $\mathcal{V}$ of t and for all correspondence $\varphi \in \prod_{i \in I} W_{\mathcal{V}}(t_i, t_{-i})$, choosing $z \in \mathcal{V}$ with $z_1 = z_2 < \frac{1}{2}$, and for each $i \in I$, $z^{'} \in \mathcal{V}$ and $y_i^{'} \in \varphi_i(z_{-i}^{'})$, we have $u_i^{\varphi}(y_i^{'}, z_{-i}^{'}) \leq 10 = u_i(z)$. Thus, this game does not have the generalized deviation property, so that Theorem 58 of Bich and Laraki [2012] cannot be applied.

This game is not (generalized) better-reply secure[6]. Indeed, consider $t = (\frac{1}{2}, \frac{1}{2}) \in X$ and $u = (10, 10)$. Then, (t, u) is in the closure of the graph of its vector function, and t is not an equilibrium. Each player i cannot obtain a payoff strictly above $u_i = 10$, for any neighborhood $\mathcal{V} \subset [0,1]$ of t_{-i} and for all well-behaved correspondence $\varphi_i : \mathcal{V} \twoheadrightarrow X_i = [0,1]$, for each $t_{-i}^{'} \in \mathcal{V}$, we then have $u_i(t_i^{'}, t_{-i}^{'}) \leq 10 = u_i$, for each $t_i^{'} \in \varphi_i(t_{-i}^{'})$. Thus, this game is not (generalized) better-reply secure. Hence Corollary 4.5 of Barelli and Soza [2009], Theorem 1 of Carmona [2011] and Theorem 3.1 in Reny [1999] cannot be applied.

This game is not weakly reciprocal upper semicontinuous[7]. Indeed, consider $t = (\frac{1}{2}, \frac{1}{2}) \in X$ and $u = (10, 10)$. Then, (t, u) is in the frontier of the graph of its vector function, and t is not an equilibrium. For each player i, we have for each $y_i \in X_i$, $u_i(y_i, \frac{1}{2}) \leq u_i = 10$. Thus, this game is not weakly reciprocal upper semicontinuous. As a results, Proposition 1 of Bagh and Jofre [2006], Theorem 4 in Prokopovych [2011] and Corollary 2 in Carmona [2009] cannot be applied.

The game is not diagonally better-reply secure. Indeed, consider $t = (\frac{1}{2}, \frac{1}{2}) \in X$ and $u = (10, 10)$. Then, (x, u) is in the closure of the graph of its vector function, and t is not an equilibrium. Player i cannot obtain a payoff strictly above u_i, for any neighborhood $\mathcal{V} \subset [0,1]$ of $\frac{1}{2}$, choosing $z_{-i} \in \mathcal{V}$ with $z_{-i} > \frac{1}{2}$. We have for each $y_i \in [0,1]$, $u_i(y_i, z_{-i}) \leq 10 = u_i$. Thus, this game is not diagonally better-reply secure, and Theorem 4.1 in Reny [1999] cannot be applied.

This game is not generalized weakly transfer continuous[8]. Indeed, consider $t = (\frac{1}{2}, \frac{1}{2}) \in X$. Each player i, for any neighborhood $\mathcal{V} \subset [0,1]^2$ of $(\frac{1}{2}, \frac{1}{2})$ and for all well-behaved correspondence $\varphi_i : \mathcal{V} \twoheadrightarrow X_i = [0,1]$, choosing $z \in \mathcal{V}$ with $z_i < z_{-i}$. Then, for each $y_i \in \varphi_1(z_i, z_{-i})$, we have $u_i(y_i, z_{-i}) \leq u_i(z_i, z_{-i}) = 10$. Thus, this game is not generalized weakly transfer continuous, so

[5] A game $G = (X_i, u_i)_{i \in I}$ has the *generalized deviation property* if whenever $\overline{x} \in X$ is not an equilibrium, there exist a neighborhood $\mathcal{V}$ of $\overline{x}$, and a correspondence $\phi_{\overline{x}} : \mathcal{V} \twoheadrightarrow X$ (having a closed graph and nonempty and convex values) so that for each $z \in \mathcal{V}$, there exists a player j for whom, $u_j^{\phi}(y_j^{'}, t_{-j}^{'}) > u_j(z)$, for each $(t^{'}, y_j^{'}) \in \text{Graph}(\phi_i)$.

[6] A game $G = (X_i, u_i)_{i \in I}$ is generalized better-reply secure if whenever $(x, u) \in \text{cl}(\Gamma)$ and x is not an equilibrium, then there exist a player i, an $\alpha > u_i$, a neighborhood $\mathcal{V}$ of x_{-i} and a well-behaved correspondence $\phi_i : \mathcal{V} \twoheadrightarrow X_i$ such that $u_i(z) \geq \alpha$, for each $z \in \text{Graph}(\phi_i)$.

[7] A game $G = (X_i,\ u_i)_{i \in I}$ is weakly reciprocal upper semicontinuous, if for any $(x, u) \in \text{Fr}(\Gamma)$, there is a player i and $\hat{x}_i \in X_i$ such that $u_i(\hat{x}_i, x_{-i}) > u_i$.

[8] A game $G = (X_i, u_i)_{i \in I}$ is said to be generalized weakly transfer continuous if whenever $x^* \in X$ is not an equilibrium, there exist a player i, a neighborhood $\mathcal{V}(x^*)$ of x^* and a well-behaved correspondence $\varphi_i : \mathcal{V}(x^*) \twoheadrightarrow X_i$ such that $\inf_{(z, y_i) \in \text{Graph}(\varphi_i)} \{u_i(y_i, z_{-i}) - u_i(z)\} > 0$.

that Theorem 3.1 of Nessah [2011] cannot be applied.

However, it is $\overline{u}$-weakly continuous secure. Indeed, let us consider in Definition 3.1

$$\overline{u}_i(y_i, x) = \sup_{\mathcal{V}\in\Omega(x)} \sup_{\varphi_i\in W_{\mathcal{V}}(y_i,x)} \inf_{(z,t_i)\in \text{Graph}(\varphi_i)} u_i(t_i, z_{-i}).$$

Let $t = (t_1, t_2)$ be a nonequilibrium strategy. If $t_1 \neq t_2$, then combining the continuity of l_i and m_i and the nonequilibrium of t, we deduce the 0-weakly continuous security of the game. If $t_1 = t_2$, by the nonequilibrium of t, we have $t_1 \geq \frac{1}{2}$. Thus, there exist a neighborhood $\mathcal{V}$ of t and a well-behaved correspondence $\phi_t : \mathcal{V} \twoheadrightarrow X$ defined by $\phi_t(x) = \{(0, 0)\}$, for each $x \in \mathcal{V}$ so that for each $z \in \mathcal{V}$ nonequilibrium, we have: if $z_1 < z_2$, let $i = 2$ then $\overline{u}_2(z_2, z) = -10 < 10 = \inf_{x\in\mathcal{V}} u_2(x_1, 0)$ and if $z_1 \geq z_2$, let $i = 1$ so $\overline{u}_1(z_1, z) = -10 < 10 = \inf_{x\in\mathcal{V}} u_1(0, x_2)$. Since for each $(x, y_j) \in \text{Graph}(\phi_{t,j})$, we have $\overline{u}_j(y_j, x) \geq \inf_{x\in\mathcal{V}} u_j(0, x_{-j})$. Then, for each $(x, y_j) \in \text{Graph}(\phi_{t,j})$, we have $\overline{u}_j(y_j, x) > 0 \geq \overline{u}_j(z_j, z)$. The game is also bounded, compact and quasiconcave, then by Theorem 3.1, the considered game possesses a Nash equilibrium.

EXAMPLE 3.4 Consider the two-player game on the square $[0, 1] \times [0, 1]$.

$$u_i(p) = \begin{cases} 4p_i & \text{if } p_{-i} \geq \frac{1}{2} \\ p_i & \text{otherwise.} \end{cases}$$

It is clear that this game is bounded, compact and quasiconcave.
This game is not continuous secure. Consider $p = (\frac{1}{2}, \frac{1}{2}) \in X$, then p is not an equilibrium. For any neighborhood $\mathcal{V}$ of p, any correspondence $\varphi : \mathcal{V} \twoheadrightarrow X$, and for each $\alpha \in \mathbb{R}^n$, we have: If for each $j = 1, 2$, $\alpha_j \leq 1$, choose $z \in \mathcal{V}$ with $z_j > \frac{1}{2}$, for $j = 1, 2$. Therefore $u_i(z) = 4z_i \geq 2 > \alpha_i$, for each $i = 1, 2$. If $\alpha_j > 1$ for some $j = 1, 2$, choose $z \in \mathcal{V}$ with $z_i < \frac{1}{2}$, for each $i = 1, 2$. Therefore for each $t_j \in \varphi_j(z)$, we have $u_j(t_j, z_{-j}) = t_j \leq 1 < \alpha_j$. Thus, this game is not continuous secure, so Theorem 2.2 and Proposition 2.4 in Barelli and Meneghel [2013] cannot be applied. Since the continuous security condition is weaker than the C-security, then Proposition 2.7 of McLennan, Monteiro and Tourky [2011] cannot be applied.

This game does not have the single lower deviation property [9]. Indeed, consider $p = (\frac{1}{2}, \frac{1}{2}) \in X$, then p is not an equilibrium. Any neighborhood $\mathcal{V}$ of p and all $y \in X$, choosing $z \in \mathcal{V}$ with $z_j > \frac{1}{2}$, for $j = 1, 2$ and for each $i \in I$, choosing $t \in \mathcal{V}$ with $t_{-i} < \frac{1}{2}$. Then, there exists a neighborhood $\mathcal{V}_z \subset (\frac{1}{2}, 1]^2$ such that $\overline{u}_i(z) \geq \inf_{z'\in\mathcal{V}_z} u_i(z_i, z'_{-i}) = 4z_i > 2 > y_i = \overline{u}_i(y_i, t_{-i})$. Thus, this game does not have the single lower deviation property, so that Theorem 2.2 in Reny [2009] cannot be applied.

This game does not have the generalized deviation property. Indeed, consider $p = (\frac{1}{2}, \frac{1}{2}) \in X$. Then, p is not an equilibrium. Any neighborhood $\mathcal{V}$ of p and for all correspondence $\varphi \in$

[9] A game $G = (X_i,\ u_i)_{i\in I}$ has the single lower deviation property if whenever x^* is not an equilibrium, there is a player a neighborhood $\mathcal{V}$ of x^* and a strategy $\overline{y} \in X$ such that for each $z \in \mathcal{V}$, there is a player j so as $\overline{u}_j(\overline{y}_j, t_{-j}) > \overline{u}_j(z)$ for all $t \in \mathcal{V}$.

$\prod_{i\in I} W_{\mathcal{V}}(p_i, p_{-i})$, choosing $z \in \mathcal{V}$ with $z = (\frac{1}{2}, \frac{1}{2})$, and for each $i \in I$, choosing $z^{'} \in \mathcal{V}$ with $z^{'}_{-i} < \frac{1}{2}$. Then, for each $y^{'}_i \in \varphi_i(z^{'}_{-i})$, we have $u^{\varphi}_i(y^{'}_i, z^{'}_{-i}) \leq u_i(y^{'}_i, z^{'}_{-i}) = y^{'}_i \leq 2 = u_i(z)$. Hence, Theorem 58 of Bich and Laraki [2012] cannot be applied.

This game is not (generalized) better-reply secure. Indeed, consider $p = (\frac{1}{2}, \frac{1}{2}) \in X$ and $u = (2, 2)$. Then, (p, u) is in the closure of the graph of its vector function, and p is not an equilibrium. Each player i cannot obtain a payoff strictly above $u_i = 2$, for any neighborhood $\mathcal{V} \subset [0, 1]$ of p_{-i} and for all well-behaved correspondence $\varphi_i : \mathcal{V} \twoheadrightarrow X_i = [0, 1]$, choosing $p^{'}_{-i} \in \mathcal{V}$ with $p^{'}_{-i} < \frac{1}{2}$, we then have $u_i(q_i, p^{'}_{-i}) = q_i \leq 2 = u_i$, for each $q_i \in \varphi_i(p^{'}_{-i})$. Thus, this game is not (generalized) better-reply secure. Therefore, Corollary 4.5 of Barelli and Soza [2009], Theorem 1 of Carmona [2011] and Theorem 3.1 in Reny [1999] cannot be applied.

This game is not generalized payoff secure. Indeed, let $i = 1$, $x = (1, \frac{1}{2})$, $\epsilon = \frac{1}{2}$. Then for any neighborhood $\mathcal{V} \subset [0, 1]$ of $\frac{1}{2}$ and for all well-behaved correspondence $\varphi_i : \mathcal{V} \twoheadrightarrow [0, 1]$, choosing $p^{'}_2 \in \mathcal{V}$ with $p^{'}_2 < \frac{1}{2}$, we then have $u_1(q_1, p^{'}_2) = q_1 \leq 1 < 4 - \epsilon$, for each $q_1 \in \varphi_1(p^{'}_2)$. Thus, this game is not generalized payoff secure, and consequently Theorem 4 in Prokopovych [2011] and Corollary 2 in Carmona [2009] cannot be applied.

However, it is $\overline{u}$-weakly continuous secure. Indeed, let us consider in Definition 3.1

$$\overline{u}_i(y_i, x) = \sup_{\mathcal{V}\in\Omega(x)} \sup_{\varphi_i\in W_{\mathcal{V}}(y_i,x)} \inf_{(z,t_i)\in \text{Graph}(\varphi_i)} \left[u_i(t_i, z_{-i}) - u_i(z)\right].$$

Let $p = (p_1, p_2)$ be a nonequilibrium strategy profile with at least one non-one coordinate. Then, there exists $i \in I$ with $p_i < 1$. Therefore, there exist a neighborhood $\mathcal{V}$ of p and $\epsilon > 0$ with $p^{'}_i + \epsilon < 1$ for all $p^{'} \in \mathcal{V}$ and a well-behaved correspondence $\phi_p : \mathcal{V} \twoheadrightarrow X$ defined by $\phi_p(p^{'}) = \{(1, 1)\}$, for each $p^{'} \in \mathcal{V}$ such that for each $z \in \mathcal{V}$, we have $\overline{u}_j(z, z_j) \leq 0$, for each j (see the proof of Proposition 4.4). For each $(t, y_i) \in \text{Graph}(\phi_p)$, we have $u_i(y_i, t_{-i}) = \begin{cases} 4, & \text{if } t_{-i} \geq \frac{1}{2} \\ 1, & \text{otherwise} \end{cases}$ and $u_i(t) = \begin{cases} 4t_i, & \text{if } t_{-i} \geq \frac{1}{2} \\ t_i, & \text{otherwise,} \end{cases}$ then $u_i(y_i, t_{-i}) - u_i(t) > 0$ (Because $t_i + \epsilon < 1$ for all $t \in \mathcal{V}$). Since for each $(t, y_i) \in \text{Graph}(\phi_{p,i})$, we have $\overline{u}_i(y_i, t) \geq \inf_{(t,y_i)\in\text{Graph}(\phi_{p,i})} [u_i(y_i, t_{-i}) - u_i(t)]$. Then, for each $(t, y_i) \in \text{Graph}(\phi_{p,i})$, we have $\overline{u}_i(y_i, t) > 0 \geq \overline{u}_i(z, z_i)$. The game is also bounded, compact and quasiconcave, so by Theorem 3.1, the considered game possesses a Nash equilibrium.

3.2 g-GENERALIZED PAYOFF SECURE

While it is simple to verify the g-weakly continuous security, it is sometimes even simpler to verify other conditions leading to it. In the following, we introduce the *g-pseudo upper semi-continuity* and the *g-generalized payoff secure* properties which are weaker conditions of *weakly reciprocal upper semicontinuous* and/or *transfer reciprocal upper semicontinuous* and generalized payoff secure, respectively and we show that under the quasiconcavity of the game, g-pseudo upper semicontinuity together with g-generalized payoff security, the game is g-weakly continuous secure.

Definition 3.2 A game $G = (X_i,\ u_i)_{i \in I}$ is said to be *g-pseudo upper semicontinuous* (PUSC) if, whenever $\overline{x} \in X$ is not an equilibrium, there exist a neighborhood $\mathcal{V}$ of $\overline{x}$ and $\epsilon > 0$ such that for each $z \in \mathcal{V}$ nonequilibrium, there exists a player j so that $\sup_{y_j \in X_j} u_j(y_j, \overline{x}_{-j}) - g_j(\overline{x}) > \overline{u}^g_j(z_j, z) + \epsilon$.

In words, a game that is g-pseudo upper semicontinuous implies that for every nonequilibrium x, for each slightly deviation z nonequilibrium of x, some player j has a strategy yielding a strictly payoff at the g-locally security level even if the others play x. The g-local security level at z means that the value of the least favorable outcome in a neighborhood of z is given by $\overline{u}^g_j(z_j, z) = \sup_{\mathcal{V}_z \in \Omega(z)} \sup_{\varphi_j \in W_{\mathcal{V}_z}(z_j, z)} \inf_{(z', z''_j) \in \text{Graph}(\varphi_j)} \left[u_j(z''_j, z'_{-j}) - g_j(z') \right]$. It is clear that G is 0-pseudo upper semicontinuous if it is upper semicontinuous. Notice that in the definition of g-pseudo upper semicontinuity, we do not employ the closure of the graph of the vector payoff function contrary to the results of Reny [1999], Bagh and Jofre [2006], Prokopovych [2011], Carmona [2009, 2011] Barelli and Soza [2009] and Barelli and Meneghel [2013].

Definition 3.3 A game $G = (X_i,\ u_i)_{i \in I}$ is said to be *g-generalized payoff secure* (g-GPS) if for all $i \in I$, $x \in X$, and $\epsilon > 0$, there exist a neighborhood $\mathcal{V}$ of x, and a well-behaved correspondence $\phi_{x,i} : \mathcal{V} \twoheadrightarrow X_i$ so that $u_i(y_i, t_{-i}) - g_i(t) \geq u_i(x) - g_i(x) - \epsilon$, for each $(t, y_i) \in \text{Graph}(\phi_{x,i})$.

If in Definition 3.3 the function $g \equiv 0$, then Definition 3.3 is identical to the definition of generalized payoff secure given by Barelli and Soza [2009]. The following notions are introduced by Bagh and Jofre [2006] and Prokopovych [2011], respectively.

Definition 3.4 A game $G = (X_i,\ u_i)_{i \in I}$ is weakly reciprocal upper semicontinuous, if for any $(x, u) \in \text{Fr}(\Gamma)$, there is a player i and $\hat{x}_i \in X_i$ such that $u_i(\hat{x}_i, x_{-i}) > u_i$.

Definition 3.5 A game $G = (X_i,\ u_i)_{i \in I}$ is transfer reciprocal upper semicontinuous, if for any $(x, u) \in \text{Fr}(\Gamma)$, and x is not a Nash equilibrium, then there is a player i and $\hat{x}_i \in X_i$ such that $u_i(\hat{x}_i, x_{-i}) > u_i$.

Proposition 3.1 *If a game $G = (X_i,\ u_i)_{i \in I}$ is weakly reciprocal upper semicontinuous at every x and x is not an equilibrium (transfer reciprocal upper semicontinuous), then it is 0-pseudo upper semicontinuous.*

Proof. Assume that G is weakly reciprocal upper semicontinuous at $\overline{x}$ ($\overline{x}$ is not an equilibrium) and it is not 0-pseudo upper semicontinuous at $\overline{x}$. Then, by definition, for each neighborhood

$\mathcal{V}$ of $\overline{x}$ and $\epsilon > 0$, there exists $z \in \mathcal{V}$, so that $\sup_{y_i \in X_i} u_i(y_i, \overline{x}_{-i}) < \overline{u}_i^0(z_i, z) + \epsilon$, for each $i \in I$. Thus, for a directed system of neighborhoods $\{\mathcal{V}^k\}_k$ of $\overline{x}$ and a sequence $\{\epsilon^k\}_k$ that converges to 0, there exists a sequence $\{z^k\}_k$ with $z^k \in \mathcal{V}^k$ so as $\{z^k\}_k$ converges to $\overline{x}$ and for each $i \in I$, we have

$$\sup_{y_i \in X_i} u_i(y_i, \overline{x}_{-i}) < \overline{u}_i^0(z_i^k, z^k) + \epsilon^k \leq u_i(z^k) + \epsilon^k.$$

Assume that $\{u(z^k)\}_k$ converges and let $\overline{u} = \lim_{k\to\infty} u(z^k)$. Then $(\overline{x}, \overline{u})$ is in the closure of the graph of G. If $(\overline{x}, \overline{u})$ is not in the frontier of G, then $\overline{u} = u(\overline{x})$ and consequently $\sup_{y_i \in X_i} u_i(y_i, \overline{x}_{-i}) \leq u_i(\overline{x})$, for each $i \in I$, which is a contradiction because $\overline{x}$ is not a Nash equilibrium. Then $(\overline{x}, \overline{u})$ is on the frontier of G. By weakly reciprocal upper semicontinuity of G, there exists a player j who has a strategy $\hat{y}_j \in X_j$ such that $u_j(\hat{y}_j, \overline{x}_{-j}) > \overline{u}_j \geq \sup_{y_j \in X_j} u_j(y_j, \overline{x}_{-j})$, which is impossible. ■

PROPOSITION 3.2 *If a game $G = (X_i,\ u_i)_{i\in I}$ is quasiconcave, g-generalized payoff secure and g-pseudo upper semicontinuous, then it is $\overline{u}^g$-weakly continuous secure, where*

$$\overline{u}_i^g(y_i, x) = \sup_{\mathcal{V}\in\Omega(x)} \sup_{\varphi_i \in W_{\mathcal{V}}(y_i,x)} \inf_{(z,t_i)\in \mathit{Graph}(\varphi_i)} \left[u_i(t_i, z_{-i}) - g_i(z)\right].$$

PROOF. Let $x \in X$ be such that it is not an equilibrium. Then, by g-pseudo upper semicontinuity of G, there exist a neighborhood $\mathcal{V}$ of x and $\epsilon > 0$ such that for each $z \in \mathcal{V}$ nonequilibrium, there exists a player $i \in I$ so that $\sup_{y_i \in X_i} u_i(y_i, x_{-i}) - g_i(x) > \overline{u}_i^g(z_i, z) + 2\epsilon$. For $\epsilon > 0$, there exists $\hat{y}$ so as $u_i(\hat{y}_i, x_{-i}) \geq \sup_{y_i \in X_i} u_i(y_i, x_{-i}) - \epsilon$. By g-generalized payoff security in $(\hat{y}_i, x_{-i})$, for $\epsilon > 0$, and $i \in I$, there exist a neighborhood $\mathcal{V}^i$ of x, and a well-behaved correspondence $\phi_{x,i} : \mathcal{V} \twoheadrightarrow X_i$ so that $u_i(y_i, t_{-i}) - g_i(t) \geq u_i(\hat{y}_i, \overline{x}_{-i}) - g_i(x) - \epsilon$, for each $(t, y_i) \in \text{Graph}(\phi_{x,i})$. We also have $\overline{u}_i^g(y_i, t) \geq \inf_{(t,y_i)\in \text{Graph}(\phi_{x,i})} [u_i(y_i, t_{-i}) - g_i(t)]$, for each $(t, y_i) \in \text{Graph}(\phi_{x,i})$. Let $\overline{\mathcal{V}} = \bigcap_{i\in I} \left(\mathcal{V}^i\right) \bigcap \mathcal{V}$ a neighborhood of x. Then for each $z \in \overline{\mathcal{V}}$ nonequilibrium, there exists a player $i \in I$ so that for each $(t, y_i) \in \text{Graph}(\phi_{x,i})$

$$\overline{u}_i^g(y_i, t) + 2\epsilon \geq u_i(\hat{y}_i, x_{-i}) - g_i(x) + \epsilon \geq \sup_{y_i \in X_i} u_i(y_i, x_{-i}) - g_i(x) > \overline{u}_i^g(z_i, z) + 2\epsilon.$$

Since the game is quasiconcave, then it is g-weakly continuous secure. ■

We have the following corollary.

COROLLARY 3.2 *If a game $G = (X_i, u_i)_{i\in I}$ is bounded, convex, compact, g-pseudo upper semicontinuous, g-generalized payoff secure, and quasiconcave, then it possesses a pure strategy Nash equilibrium.*

REMARK 3.5 If $g \equiv u$, then the u-pseudo upper semicontinuity is automatically satisfied. Indeed, since $\overline{u}_j^u(z_j, z) \leq 0$ (see the proof of Proposition 4.4) and $\overline{x}$ is not an equilibrium, then there is a neighborhood $\mathcal{V}$ of $\overline{x}$ and $\epsilon > 0$ such that for each $z \in \mathcal{V}$ nonequilibrium, there exists a player j so that $\sup_{y_j \in X_j} u_j(y_j, \overline{x}_{-j}) - u_j(\overline{x}) > \epsilon \geq \overline{u}_j^u(z_j, z) + \epsilon$. In this case, the u-generalized payoff secure becomes the generalized weakly transfer continuity introduced by Nessah [2011] (see Subsection 4.3).

If $g \equiv 0$, then we obtain the following result.

COROLLARY 3.3 *If a game $G = (X_i, u_i)_{i \in I}$ is bounded, convex, compact, 0-pseudo upper semicontinuous, generalized payoff secure, and quasiconcave, then it possesses a pure strategy Nash equilibrium.*

Proposition 3.2 and the following example show that Corollary 3.3 generalizes strictly Proposition 1 of Bagh and Jofre [2006], Corollary 3.3-3.4 of Reny [1999] and Corollary 8.5 of McLennan, Monteiro and Tourky [2011].

EXAMPLE 3.5 Recall the following example considered in Example 3.4. For player $i = 1, 2$ and $t = (t_1, t_2) \in X = [0, 1]^2$, let the payoff functions for the players be given by

$$u_i(t_1, t_2) = \begin{cases} l_i(t) = 10, & \text{if } t_i < t_{-i} \\ k_i(t), & \text{if } t_i = t_{-i} = t \\ m_i(t_{-i}) = -10, & \text{if } t_i > t_{-i}, \end{cases}$$

where $k_i(t) = 10$ when $t < \frac{1}{2}$ and $k_i(t) = 0$ when $t \geq \frac{1}{2}$. This game does not have the generalized deviation property, it is not (generalized) better-reply secure. Furthermore, the game is not diagonally better-reply secure (see Example 3.2). Indeed, consider $t = (\frac{1}{2}, \frac{1}{2}) \in X$ and $u = (10, 10)$. Then, (x, u) is in the closure of the graph of its vector function, and t is not an equilibrium. Player i cannot obtain a payoff strictly above u_i, for any neighborhood $\mathcal{V} \subset [0, 1]$ of $\frac{1}{2}$, choosing $z \in \mathcal{V}$ with $z_{-i} > \frac{1}{2}$. We have for each $y_i \in [0, 1]$, $u_i(y_i, z_{-i}) \leq 10 = u_i$. Thus, this game is not diagonally better-reply secure, so that Theorem 4.1 in Reny [1999] cannot be applied.

Finally, this game is not weakly reciprocal upper semicontinuous. Indeed, consider $t = (\frac{1}{2}, \frac{1}{2}) \in X$ and $u = (10, 10)$. Then, (t, u) is in the frontier of the graph of its vector function, and t is not an equilibrium. For each player i, we have for each $y_i \in X_i$, $u_i(y_i, \frac{1}{2}) \leq u_i = 10$. Thus, this game is not weakly reciprocal upper semicontinuous. Then, Proposition 1 of Bagh and Jofre [2006], Theorem 4 in Prokopovych [2011] and Corollary 8.5 of McLennan, Monteiro and Tourky [2011] cannot be applied.

However, it is 0-pseudo upper semicontinuous. Indeed, let $t = (t_1, t_2)$ be a nonequilibrium strategy. If $t_1 \neq t_2$, then combining the continuity of l_i and m_i and the nonequilibrium of t, we deduce the pseudo upper semicontinuous. If $t_1 = t_2$, by the nonequilibrium of t, we have $t_1 \geq \frac{1}{2}$. Thus, there exist a strategy $\overline{y} = (0, 0)$, $\epsilon = 1$ and a neighborhood $\mathcal{V}$ of t so that for each $z \in \mathcal{V}(t)$, if $z_1 > z_2$, let $i = 2$ and for each $\mathcal{V}_z$ neighborhood of z, there exists $z^{'} \in \mathcal{V}_z$ with $z_1^{'} < z_2$ so $u_2(z_1^{'}, z_2) = -10 < 10 - \epsilon = u_2(t_1, \overline{y}_2) - \epsilon$. If $z_1 \leq z_2$, let $i = 1$ and for each $\mathcal{V}_z$ neighborhood of z, there exists $z^{'} \in \mathcal{V}_z$ with $z_2^{'} < z_1$ so $u_1(z_1, z_2^{'}) = -10 < 10 - \epsilon = u_1(\overline{y}_1, t_2) - \epsilon$. Since the game is also bounded, compact, quasiconcave and generalized payoff secure, it implies that by Corollary 3.3, the considered game possesses a Nash equilibrium.

4 RELATED RESULTS

In this section, we show that the g-weakly continuous security is weaker than the most important results about the existence of pure strategy equilibrium [10]. Theorem 3.1 in Reny [1999] shows that a game $G = (X_i, u_i)_{i \in I}$ possesses a Nash equilibrium if it is compact, quasiconcave and better-reply secure. Reny [1999] and Bagh and Jofre [2006] provided sufficient conditions for a game to be better-reply secure. Reny [1999] showed that a game $G = (X_i, u_i)_{i \in I}$ is better-reply secure if it is payoff secure and reciprocal upper semicontinuous. Bagh and Jofre [2006] further showed that $G = (X_i, u_i)_{i \in I}$ is better-reply secure if it is payoff secure and weakly reciprocal upper semicontinuous. Morgan and Scalzo [2007] showed that $G = (X_i, u_i)_{i \in I}$ is better-reply secure if u_i is pseudocontinuous, $\forall i \in I$. Prokopovych [2011] proved that if a game $G = (X_i, u_i)_{i \in I}$ is payoff secure then G is better-reply secure if and only if it is transfer reciprocal upper semicontinuous. Corollary 2 in Carmona [2009] shows that a $G = (X_i,\ u_i)_{i \in I}$ possesses a Nash equilibrium if it is compact, quasiconcave, weakly upper semicontinuous and weakly payoff secure. Barelli and Soza [2009] show that if $G = (X_i, u_i)_{i \in I}$ is compact, quasiconcave and generalized better-reply secure, then it has a Nash equilibrium. Theorem 1 in Carmona [2011] shows that a $G = (X_i,\ u_i)_{i \in I}$ possesses a Nash equilibrium if it is compact, quasiconcave, weakly better-reply secure. Carmona [2011] shows in Theorems 2-3 that generalized better-reply security is equivalent to weak better-reply security. Nessah [2011] shows that a $G = (X_i,\ u_i)_{i \in I}$ possesses a Nash equilibrium if it is compact, quasiconcave and generalized weakly transfer continuous. Barelli and Meneghel [2013] show that if G is compact, convex and continuous secure, then it has a Nash equilibrium.

In the following subsections, we show that depending on the choice of the function $\overline{u}$, we show that the weakly continuous security generalizes most of the notions of continuity introduced in the literature.

[10]Such as the *continuous security* of Barelli and Meneghel [2013], *C-secure* of McLennan, Monteiro and Tourky [2011], *generalized deviation property* of Bich and Laraki [2012], *generalized better-reply security* of Barelli and Meneghel [2013], Carmona [2011], Barelli and Soza [2009], *lower single deviation property* of Reny [2009], *better-reply security* of Reny [1999], *generalized weakly transfer continuity* of Nessah [2011] and the results of Prokopovych [2011, 2012] and Carmona [2009].

4.1 0-WEAKLY CONTINUOUS SECURE

Let us consider in Definition 3.1, $\overline{u}_i^0(y_i, x) = \sup_{\mathcal{V}\in\Omega(x)} \sup_{\varphi_i\in W_{\mathcal{V}}(y_i,x)} \inf_{(z,t_i)\in \text{Graph}(\varphi_i)} u_i(t_i, z_{-i})$. Then we obtain the following definition.

DEFINITION 4.1 A game $G = (X_i, u_i)_{i\in I}$ is said to be 0-*weakly continuous secure* (0WCS) if whenever $\overline{x} \in X$ is not an equilibrium, there exist a neighborhood $\mathcal{V}$ of $\overline{x}$ and a well-behaved correspondence $\phi_{\overline{x}} : \mathcal{V} \twoheadrightarrow X$ so that for each $z \in \mathcal{V}$ nonequilibrium, there exists a player j such that for each $(x, y_j) \in \text{Graph}(\phi_{\overline{x},j})$, we have

$$z_j \notin co\{t_j \in X_j \text{ such that } \overline{u}_j^0(t_j, z) \geq \overline{u}_j^0(y_j, x)\}.$$

We have the following proposition.

PROPOSITION 4.1 *If the game G is continuous secure, then it is* 0*-weakly continuous secure.*

PROOF. Suppose that G is continuous secure at x where x is not an equilibrium. Then there is a neighborhood $\mathcal{V}$ of x, $\alpha \in \mathbb{R}^n$, and a well-behaved[11] correspondence $\phi_x : \mathcal{V} \twoheadrightarrow X$ so that

(1) for each $t \in \mathcal{V}$, and $i \in I$, we have $\phi_{x,i}(t) \subseteq B_i(t, \alpha_i)$, and

(2) for each $z \in \mathcal{V}$, there exists a player j for whom, $z_j \notin coB_j(z, \alpha_j)$.

Where $B_i(x, \alpha_i) = \{y_i \in X_i \text{ such that } u_i(y_i, x_{-i}) \geq \alpha_i\}$.

Condition (1) implies that for each $i \in I$, $t \in \mathcal{V}$ and $y_i \in \phi_{x,i}(t)$, $u_i(y_i, t_{-i}) \geq \alpha_i$. Therefore, $\inf_{(t,y_i)\in \text{Graph}(\phi_{x,i})} u_i(t_i, z_{-i}) \geq \alpha_i$. We have for each $(t, y_i) \in \text{Graph}(\phi_{x,i})$,

$$\overline{u}_i^0(y_i, t) \geq \inf_{(t,y_i)\in \text{Graph}(\phi_{x,i})} u_i(t_i, z_{-i}) \geq \alpha_i.$$

Assume that for some $\tilde{z} \in \mathcal{V}$ nonequilibrium, and for each player $i \in I$ so that

$$\tilde{z}_i \in co\{t_i \in X_i \text{ such that } \overline{u}_i^0(t_i, \tilde{z}) \geq \overline{u}_i^0(\tilde{y}_i, \tilde{x}), \text{ for each } (\tilde{x}, \tilde{y}_j) \in \text{Graph}(\phi_{x,i})\}.$$

There is a finite set $A_i = \{t_i^1, ..., t_i^{p_i}\} \subseteq co\{t_i \in X_i \text{ such that } \overline{u}_i^0(t_i, \tilde{z}) \geq \overline{u}_i^0(\tilde{y}_i, \tilde{x}), \text{ for each } (\tilde{x}, \tilde{y}_j) \in \text{Graph}(\phi_{x,i})\}$ so that $\tilde{z}_i \in coA_i$ for each $i \in I$. Therefore, for each $i \in I$, and $h = 1, ..., p_i$, we have

$$\overline{u}_i^0(t_i^h, \tilde{z}) \geq \overline{u}_i^0(\tilde{y}_i, \tilde{x}) \geq \alpha_i.$$

[11] As mentioned by Carmona and Podczeck [2012], the definition of *continuously security* in Barelli and Meneghel [2013] is not exactly equal to the one presented here. Barelli and Meneghel [2013] do not require well-behaved correspondence. Unfortunately, the proof of Theorem 2.2 in Barelli and Meneghel [2013] does not go through without well-behaved correspondence.

By condition (2), for $\tilde{z} \in \mathcal{V}$, there exists a player $j \in I$ so as $\tilde{z}_j \notin coB_j(\tilde{z}, \alpha_j)$. If $A_j \subseteq B_j(\tilde{z}, \alpha_j)$, then we obtain a contradiction $\tilde{z}_j \in coA_j \subseteq coB_j(\tilde{z}, \alpha_j)$. Thus $A_j \nsubseteq B_j(\tilde{z}, \alpha_j)$ and therefore there is $h_0 = 1, ..., p_j$ such that $t_j^{h_0} \notin B_j(\tilde{z}, \alpha_j)$, i.e. $\alpha_j > u_j(t_j^{h_0}, \tilde{z}_{-j})$. Let $\epsilon = \frac{1}{2}[\alpha_j - u_j(t_j^{h_0}, \tilde{z}_{-j})] > 0$. Then for $\epsilon > 0$, there exists a neighborhood $\mathcal{V}_{\tilde{z}}$ so that

$$u_j(t_j^{h_0}, \tilde{z}_{-j}) \geq \alpha_j - \epsilon = \frac{1}{2}\alpha_j + \frac{1}{2}u_j(t_j^{h_0}, \tilde{z}_{-j}).$$

Thus, $u_j(t_j^{h_0}, \tilde{z}_{-j}) \geq \alpha_j > u_j(t_j^{h_0}, \tilde{z}_{-j})$ which is a contradiction. ■

Consequently, we deduce the following corollaries.

COROLLARY 4.1 *(Barelli and Meneghel [2013]) If $G = (X_i, u_i)_{i \in I}$ is compact, convex and continuous secure, then it has a pure strategy Nash equilibrium.*

COROLLARY 4.2 *(Barelli and Meneghel [2013]) If $G = (X_i, u_i)_{i \in I}$ is compact, quasiconcave and generalized better-reply secure, then it has a pure strategy Nash equilibrium.*

PROPOSITION 4.2 *If the game G is C-secure, then it is 0-weakly continuous secure.*

PROOF. The continuous security of Barelli and Meneghel [2013] is weaker than the C-security condition of McLennan, Monteiro and Tourky [2011]. Then by Proposition 4.1, the considered game is 0-weakly continuous secure ■

COROLLARY 4.3 *(McLennan, Monteiro and Tourky [2011]) If $G = (X_i, u_i)_{i \in I}$ is compact, convex and C-secure, then it has a pure strategy Nash equilibrium.*

Since Theorem 2.2 in Barelli and Meneghel [2013] generalizes all the results of Reny [1999], Bagh and Jofre [2006], Prokopovych [2011], Morgan and Scalzo [2007], Carmona [2009, 2011] and Barelli and Soza [2009], then by Proposition 4.1 and Examples 3.3-3.4, we have the following corollary:

COROLLARY 4.4 *Theorem 3.1 is strictly weaker than the results of Barelli and Meneghel [2013], Bich and Laraki [2012], Reny [1999], Bagh and Jofre [2006], Prokopovych [2011], Prokopovych [2012], Morgan and Scalzo [2007], Carmona [2009, 2011], Barelli and Soza [2009] and Nishimura and Friedman [1981].*

PROPOSITION 4.3 *If the game G is quasiconcave and has the lower single deviation property, then it is 0-weakly continuous secure. Therefore, Theorem 3.1 generalizes Theorem 2.2 in Reny [2009].*

PROOF. Assume that the game G has the lower single deviation property and it is not 0-weakly continuous secure. Then for $x \in X$ nonequilibrium, there exist a neighborhood $\mathcal{V}$ of x and a strategy $y \in X$ so that for each $z \in \mathcal{V}$, there is a player j such as for each $t \in \mathcal{V}$.

$$\overline{u}_j^0(y_j, t) > \overline{u}_j^0(z_j, z). \tag{4.1}$$

Let $\mathcal{U}_x$ be an open neighborhood of x so that $\text{cl}(\mathcal{U}_x) \subset \mathcal{V}$. Since G is not 0-weakly continuous secure, then for neighborhood $\mathcal{U}_x$, and $\phi_x : \mathcal{U}_x \twoheadrightarrow X$ defined by $\phi_x(t) = \{y\}$, there exists $\tilde{z} \in \mathcal{U}_x$ so that

$$\inf_{t \in \text{cl}(\mathcal{U}_x)} \overline{u}_i^0(y_i, t) \leq \inf_{t \in \mathcal{U}_x} \overline{u}_i^0(y_i, t) \leq \overline{u}_i^0(\tilde{z}_i, \tilde{z}), \text{ for each } i \in I. \tag{4.2}$$

Let $z = \tilde{z}$ in (4.1), then there is a player j so that

$$\overline{u}_j^0(y_j, t) > \overline{u}_j^0(\tilde{z}_j, \tilde{z}), \text{ for each } t \in \mathcal{V}. \tag{4.3}$$

The function $t \mapsto \overline{u}_j^0(y_j, t)$ is lower semicontinuous on the compact $\text{cl}(\mathcal{U}_x)$, then there is $\tilde{t} \in \text{cl}(\mathcal{U}_x)$ so that

$$\overline{u}_j^0(y_j, \tilde{t}) = \inf_{t \in \text{cl}(\mathcal{U}_x)} \overline{u}_j^0(y_j, t) \leq \overline{u}_j^0(\tilde{z}_j, \tilde{z}). \tag{4.4}$$

So inequalities (4.3) and (4.4) implies that $\overline{u}_j^0(y_j, \tilde{t}) \leq \overline{u}_j^0(\tilde{z}_j, \tilde{z}) < \overline{u}_j^0(y_j, t)$ which is impossible for $t = \tilde{t} \in \text{cl}(\mathcal{U}_x) \subset \mathcal{V}$. ■

COROLLARY 4.5 *(Reny [2009]) If $G = (X_i, u_i)_{i \in I}$ is bounded, compact, quasiconcave and has the lower single-deviation property, then it has a pure strategy Nash equilibrium.*

4.2 u-WEAKLY CONTINUOUS SECURE

Let us consider in Definition 3.1, $\overline{u}_i^u(y_i, x) = \sup\limits_{\mathcal{V} \in \Omega(x)} \sup\limits_{\varphi_i \in W_{\mathcal{V}}(y_i, x)} \inf\limits_{(z, t_i) \in \text{Graph}(\varphi_i)} [u_i(t_i, z_{-i}) - u_i(z)]$. Then we obtain the following definition.

DEFINITION 4.2 A game $G = (X_i, u_i)_{i \in I}$ is said to be u-*weakly continuous secure* (uWCS) if whenever $\overline{x} \in X$ is not an equilibrium, there exist a neighborhood $\mathcal{V}$ of $\overline{x}$ and a well-behaved correspondence $\phi_{\overline{x}} : \mathcal{V} \twoheadrightarrow X$ so that for each $z \in \mathcal{V}$ nonequilibrium, there exists a player j such that for each $(x, y_j) \in \text{Graph}(\phi_{\overline{x}, j})$, we have

$$z_j \notin co\{t_j \in X_j \text{ such that } \overline{u}_j^u(t_j, z) \geq \overline{u}_j^u(y_j, x)\}.$$

REMARK 4.1 If $G = (X_i, u_i)_{i \in I}$ is quasiconcave, then the function $\overline{u}_i^u(y_i, x)$ is quasiconcave in y_i, for each x (see Nessah [2011]). Thus the condition $z_j \notin co\{t_j \in X_j$ such that $\overline{u}_j^u(t_j, z) \geq \overline{u}_j^u(y_j, x)$ for each $(x, y_j) \in \text{Graph}(\phi_{\overline{x}, j})\}$ in Definition 4.3 becomes $\overline{u}_j^u(y_j, x) > \overline{u}_j^u(z_j, z)$, for each $(x, y_j) \in \text{Graph}(\phi_{\overline{x}, j})$.

Recall the definition of generalized weakly transfer continuity: A game $G = (X_i, u_i)_{i\in I}$ is said to be *generalized weakly transfer continuous* if whenever $x^* \in X$ is not an equilibrium, there exist a player i, a neighborhood $\mathcal{V}$ of x^* and a well-behaved correspondence $\varphi_i : \mathcal{V} \twoheadrightarrow X_i$ such that $\inf\limits_{(z,y_i)\in \text{Graph}(\varphi_i)} [u_i(y_i, z_{-i}) - u_i(z)] > 0$.

We have the following proposition.

PROPOSITION 4.4 *If the game G is quasiconcave and generalized weakly transfer continuous, then it is u-weakly continuous secure.*

PROOF. For each $x \in X$, we have $\overline{u}^u_j(x_j, x) \leq 0$. Indeed, if $\overline{u}^u_j(x_j, x) > 0$ for some $i \in I$ and $x \in X$, choose $\epsilon > 0$ such that $\overline{u}^u_i(x, x_i) > 2\epsilon$. Then there exists a neighborhood $\mathcal{V}$ of x and a well-behaved correspondence $\phi : \mathcal{V} \twoheadrightarrow X_i$ with $x_i \in \phi_i(x)$ such that for each $(z, y_i) \in \text{Graph}(\phi_i)$, we have $u_i(y_i, z_{-i}) > u_i(z_i, z_{-i}) + \epsilon$. Then for $z = x$ and $y_i = x_i$, we obtain $u_i(x) > u_i(x) + \epsilon$, *i.e.* $\epsilon < 0$ which is impossible.

Now, assume that G is generalized weakly transfer continuous. Then if $x \in X$ is not an equilibrium, there exist a player i, a neighborhood $\mathcal{V}$ of x and a well-behaved correspondence $\varphi_i : \mathcal{V} \twoheadrightarrow X_i$ such that $\inf\limits_{(x',y'_i)\in \text{Graph}(\varphi_i)} \left[u_i(y'_i, x'_{-i}) - u_i(x')\right] > 0$. Thus for each $z \in \mathcal{V}(x)$, there exists a player $j = i$ for whom, $\overline{u}^u_j(y_j, t) \geq \inf\limits_{(x',y'_j)\in \text{Graph}(\varphi_j)} \left[u_j(y'_j, x'_{-j}) - u_j(x')\right] > 0 \geq \overline{u}^u_j(z_j, z)$, for each $(t, y_j) \in \text{Graph}(\phi_{x,j})$. ■

Consequently, we have the following corollary.

COROLLARY 4.6 *(Nessah [2011]) If $G = (X_i, u_i)_{i\in I}$ is bounded, compact, quasiconcave and generalized weakly transfer continuous, then it has a pure strategy Nash equilibrium.*

4.3 $\underline{u}$-WEAKLY CONTINUOUS SECURE

Let us consider in Definition 3.1, $\overline{u}^{\underline{u}}_i(y_i, x) = \sup\limits_{\mathcal{V}\in\Omega(x)} \inf\limits_{z\in\mathcal{V}} [u_i(y_i, z_{-i}) - \underline{u}_i(z)]$ where $\underline{u}_i(z) = \sup\limits_{\mathcal{V}\in\Omega(z)} \inf\limits_{z'\in\mathcal{V}} u_i(z_i, z'_{-i})$. Then we obtain the following definition.

DEFINITION 4.3 A game $G = (X_i, u_i)_{i\in I}$ is said to be $\underline{u}$*-weakly continuous secure* ($\underline{u}$WCS) if whenever $\overline{x} \in X$ is not an equilibrium, there exist a neighborhood $\mathcal{V}$ of $\overline{x}$ and a well-behaved correspondence $\phi_{\overline{x}} : \mathcal{V} \twoheadrightarrow X$ so that for each $z \in \mathcal{V}$ nonequilibrium, there exists a player j such that for each $(x, y_j) \in \text{Graph}(\phi_{\overline{x},j})$, we have

$$z_j \notin co\{t_j \in X_j \text{ such that } \overline{u}^{\underline{u}}_j(t_j, z) \geq \overline{u}^{\underline{u}}_j(y_j, x)\}.$$

REMARK 4.2 For each $x \in X$, $\overline{u}^{\underline{u}}_i(x_i, x) \leq 0$. Indeed, if $\overline{u}^{\underline{u}}_i(x, x_i) > 0$ for some $i \in I$ and $x \in X$. Choose $\epsilon > 0$ with $\overline{u}^{\underline{u}}_i(x, x_i) > 2\epsilon$, then there exists a neighborhood $\mathcal{V}$ of x such that for each $z \in \mathcal{V}$, we have $u_i(x_i, z_{-i}) > \underline{u}_i(z) + \epsilon$. Let $z = (x_i, z_{-i}) \in \mathcal{V}$, thus $u_i(x_i, z_{-i}) >$

$\sup_{\mathcal{V}_z} \inf_{z' \in \mathcal{V}_z} u_i(x_i, z'_{-i}) + \epsilon$. Then, for $\mathcal{V}_z = \mathcal{V}$, we have $u_i(x_i, z_{-i}) > \inf_{z' \in \mathcal{V}} u_i(x_i, z'_{-i}) + \epsilon$, for each $z = (x_i, z_{-i}) \in \mathcal{V}$. By definition of $\inf$, for $\epsilon > 0$ there exists $\tilde{z} \in \mathcal{V}$ such that $u_i(x_i, \tilde{z}_{-i}) \leq \inf_{z' \in \mathcal{V}} u_i(x_i, z'_{-i}) + \frac{\epsilon}{2}$. Thus, we obtain $\inf_{z' \in \mathcal{V}} u_i(x_i, z'_{-i}) + \epsilon < u_i(x_i, \tilde{z}_{-i}) \leq \inf_{z' \in \mathcal{V}} u_i(x_i, z'_{-i}) + \frac{\epsilon}{2}$; *i.e.* $\epsilon < 0$ which is a contradiction.

Nessah and Tian [2011]: A game $G = (X_i, u_i)_{i \in I}$ is said to be *quasi-weakly transfer continuous* if whenever $x^* \in X$ is not an equilibrium, there exists a player i, $\overline{y}_i \in X_i$, $\epsilon > 0$, and some neighborhood $\mathcal{V}$ of x^* such that for every $z \in \mathcal{V}$ and every neighborhood $\mathcal{V}_z$ of z, $u_i(\overline{y}_i, z_{-i}) > u_i(z_i, z'_{-i}) + \epsilon$ for some $z' \in \mathcal{V}_z$.

We have the following proposition.

PROPOSITION 4.5 *If the game G is quasiconcave and quasi-weakly transfer continuous, then it is $\overline{u}$-weakly continuous secure.*

PROOF. For each $x \in X$, we have $\overline{u}^{\overline{u}}_j(x_j, x) \leq 0$. Now, assume that G is quasi-weakly transfer continuous. Then if $x \in X$ is not an equilibrium, there exist a player i, a neighborhood $\mathcal{V}$ of x, $\epsilon > 0$ and a strategy $y_i \in X_i$ such that $u_i(y_i, t_{-i}) > \overline{u}_i(t) + \epsilon$, for each $t \in \mathcal{V}$. Let us consider the following well-behaved correspondence $\phi_x : \mathcal{V} \twoheadrightarrow X$ defined by $\phi_x(z) = \{(y_1, y_2)\}$. Since for each $(t, y'_j) \in \text{Graph}(\phi_{x,j})$, we have $\overline{u}^{\overline{u}}_j(y'_j, t) \geq \inf_{(t, y'_j) \in \text{Graph}(\phi_{x,j})} [u_j(y'_j, t_{-j}) - \overline{u}_j(t)]$. Thus for each $z \in \mathcal{V}$ nonequilibrium, there exists a player $j = i$ for whom, $\overline{u}^{\overline{u}}_j(y'_j, t) > 0 \geq \overline{u}^{\overline{u}}_j(z_j, z)$, for each $(t, y'_j) \in \text{Graph}(\phi_{x,j})$. ■

Then, we have the following corollary.

COROLLARY 4.7 *(Nessah and Tian [2011]) If $G = (X_i, u_i)_{i \in I}$ is bounded, compact, quasiconcave and quasi-weakly transfer continuous, then it has a pure strategy Nash equilibrium.*

5 APPLICATIONS

In this section, we give some game examples that do not satisfy the conditions of existing theorems. Examples 5.1-5.3 are games that have pure strategy equilibria by our Theorem, but which violate the continuity and/or quasiconcavity conditions. Finally, Example 5.4 is an application to shared resource games with characterization existence.

EXAMPLE 5.1 Let us consider the following two-player games of complete information studied by Baye, Kovenock and de Vries [2012] in which each player $i = 1, 2$ chooses an action x_i from the strategy space $X_i = [0, A]$ (A is a constant with $A > -\frac{v+\gamma}{\alpha+\theta-\beta-\delta}$) and where payoffs are

$$u_i(x) = \begin{cases} W(x) = v - \beta x_i - \delta x_{-i}, & \text{if } x_i > x_{-i} \\ L(x) = -\gamma - \alpha x_i - \theta x_{-i}, & \text{if } x_i < x_{-i} \\ T(x) = \frac{1}{2}[L(x) + W(x)], & \text{if } x_i = x_{-i}. \end{cases}$$

Suppose that $v + \gamma > 0$. This game can be interpreted as contests with rank-order spillovers. δ and θ parameters capture the externalities that contestants may inflict on each other. α and β refer to first and second order direct effects.

Baye, Kovenock and de Vries [2012] proved that this game has a pure strategy Nash equilibrium if and only if $\alpha \leq 0$, $\beta \geq 0$ and $\alpha + \theta - \beta - \delta < 0$. Under conditions $\alpha \leq 0$, $\beta \geq 0$, the considered game is quasiconcave. Let us show that it is 0-weakly continuous secure. Let $x = (x_1, x_2) \in X$ be nonequilibrium. If $x_1 \neq x_2$, then by continuity of L, or W, and the nonequilibrium of x, the game is 0-weakly continuous secure at x. If $x_1 = x_2 = r$, let $M(z) = \max(L(z), W(z))$. Let $\mathcal{V}$ an open neighborhood of (r, r) not containing an equilibrium. Then, by condition $\alpha + \theta - \beta - \delta < 0$ and $\mathcal{V}$ do not containing an equilibrium, we have $\inf\limits_{t \in \mathcal{V}} M(t) > \sup\limits_{t \in \mathcal{V}} \frac{1}{2}[L(t) + W(t)]$. By continuity of M (Berge Maximum Theorem), there exists a continuous function $\tau : \mathcal{V} \to \mathcal{V}$ so that for each $t \in \mathcal{V}$, we have $u_i(\tau_i(t), t_{-i}) \geq M(t)$, for each $i = 1, 2$. Let us consider the following well-behaved correspondence $\phi_x : \mathcal{V} \twoheadrightarrow \mathcal{V}$ defined by $\phi_x(z) = \{(\tau_1(z), \tau_2(z))\}$, for each $z \in \mathcal{V}$. Therefore, $\inf\limits_{(t, y_i) \in \text{Graph}(\phi_{x,i})} u_i(y_i, t_{-i}) = \inf\limits_{t \in \mathcal{V}} M(\tau_i(t), t_{-i}) \geq \inf\limits_{t \in \mathcal{V}} M(t_i, t_{-i})$. We have also for each $z \in \mathcal{V}$ nonequilibrium, there exist a player $j = 1, 2$ so that $\overline{u}_j^0(z_j, z) \leq \frac{1}{2}[L(z) + W(z)] < \inf\limits_{t \in \mathcal{V}} M(t) \leq \inf\limits_{(t, y_i) \in \text{Graph}(\phi_{x,i})} u_i(y_i, t_{-i})$[12]. Since the game is also bounded, compact and quasiconcave, so by Theorem 3.1, the considered game possesses a Nash equilibrium.

EXAMPLE 5.2 Consider the two-player game studied by Baye, Tian and Zhou [1993]. Two duopolists have zero costs and set prices (p_1, p_2) on $X = [0, T] \times [0, T]$. The payoff functions are (for $0 < c < T$):

$$u_i(p) = \begin{cases} p_i, & \text{if } p_i \leq p_{-i} \\ p_i - c, & \text{if } p_i > p_{-i}. \end{cases}$$

We can interpret the game as a duopoly in which each firm has committed to pay brand loyal consumers a penalty of c if the other firm beats its price. These payoffs are not quasiconcave nor continuous. However, the game is 0-weakly continuous secure.

Let $x = (x_1, x_2) \in X$ be nonequilibrium. If $x_1 \neq x_2$, then by continuity of u_i, and the nonequilibrium of x, the game is 0-weakly continuous secure at x. If $x_1 = x_2 = r$, and since $x = (r, r)$ is not an equilibrium, then $r < T - c$. Let $\mathcal{V} \subset [0, T - c) \times [0, T - c)$ an open neighborhood of (r, r) not containing an equilibrium and a well-behaved correspondence $\phi_x : \mathcal{V} \twoheadrightarrow \mathcal{V}$ defined by $\phi_x(z) = \{(T, T)\}$, for each $z \in \mathcal{V}$. Therefore, $\overline{u}_i^0(y_i, t) \geq \alpha = T - c$, for each $(t, y_i) \in \text{Graph}(\phi_{x,i})$. We have $B_j(z, \alpha) = \{t_j \in X_j \text{ such that } \overline{u}_j^0(t_j, z) \geq \alpha\} = \{t_j \in [0, 1] \text{ so that } t_j \geq T - c\}$. Thus for each $z \in \mathcal{V}$, $z_j \notin coB_j(z, \alpha)$ for each $j = 1, 2$. Since the

[12]If $z_i \neq z_{-i}$, then there is a player j so that $\overline{u}_j^0(z_j, z) \leq u_j(z) = \min(W(z), L(z))$ and if $z_i = z_{-i}$, then there is a player j so that $\overline{u}_j^0(z_j, z) \leq \frac{1}{2}[L(z) + W(z)]$

game is also bounded, compact and convex, so by Theorem 3.1, the considered game possesses a Nash equilibrium.

EXAMPLE 5.3 Consider the following silent duel games (see Karlin [1959]). The payoff functions are:

$$u_1(x_1, x_2) = \begin{cases} 2x_1 - 1, & \text{if } x_1 < x_2 \\ 0, & \text{if } x_1 = x_2 \\ 1 - 2x_2, & \text{if } x_1 > x_2 \end{cases}$$

and $u_2(x) = -u_1(x)$.

In this game, the payoff function $u_i(x_1, x_2)$ is not quasiconcave in y_i for $i = 2$. If $x_1 \neq x_2$, then by continuity of u_i, and the nonequilibrium of x, the game is 0-weakly continuous secure at x. If $x_1 = x_2 = r$ and since $x = (r, r)$ is not an equilibrium, then $r \neq \frac{1}{2}$. For some $\epsilon > 0$, let $\mathcal{V} \subset (r - \epsilon, 1] \times (r - \epsilon, 1]$ if $r - \epsilon > \frac{1}{2}$ and $\mathcal{V} \subset [0, r + \epsilon) \times [0, r + \epsilon)$ if $r + \epsilon < \frac{1}{2}$ an open neighborhood of (r, r) not containing an equilibrium and a well-behaved correspondence $\phi_x : \mathcal{V} \twoheadrightarrow \mathcal{V}$ defined by for each $z \in \mathcal{V}$, $\phi_x(z) = \begin{cases} \{(r - \epsilon, r - \epsilon)\}, & \text{if } r - \epsilon > \frac{1}{2} \\ \{(r + \epsilon, r + \epsilon)\}, & \text{if } r + \epsilon < \frac{1}{2}. \end{cases}$ Therefore, there is $\alpha = \begin{cases} \{(2(r - \epsilon) - 1, 2(r - \epsilon) - 1)\}, & \text{if } r - \epsilon > \frac{1}{2} \\ \{(1 - 2(r + \epsilon), 1 - 2(r + \epsilon))\}, & \text{if } r + \epsilon < \frac{1}{2} \end{cases}$ so that $\overline{u}_i^0(y_i, t) \geq \alpha_i$, for each $(t, y_i) \in \text{Graph}(\phi_{x,i})$ and $i = 1, 2$. Let $B_j^0(z, \alpha_j) = \{t_j \in X_j \text{ such that } \overline{u}_j^0(t_j, z) \geq \alpha_j\}$.

(1) If $r - \epsilon > \frac{1}{2}$. For each $z \in \mathcal{V}$, if $z_1 \geq z_2$, then $z_1 \notin coB_1^0(z, \alpha_1)$ and if $z_1 < z_2$, then $z_2 \notin coB_2^0(z, \alpha_2)$.

(2) If $r + \epsilon < \frac{1}{2}$. For each $z \in \mathcal{V}$, if $z_1 \geq z_2$, then $z_2 \notin coB_2^0(z, \alpha_2)$ and if $z_1 < z_2$, then $z_1 \notin coB_1^0(z, \alpha_1)$.

Since the game is also bounded, compact and convex, it follows by Theorem 3.1 that the game possesses a pure strategy Nash equilibrium.

EXAMPLE 5.4 Rothstein [2007] studies a class of shared resource games with discontinuous payoffs, which includes a wide class of games such as the canonical game of fiscal competition for mobile capital. In these games, players compete for a share of a resource that is in fixed total supply, except perhaps at certain joint strategies. Each player's payoff depends on her opponents' strategies only through the effect those strategies have on the amount of the shared resource that the player obtains.

Formally, for such a game $G = (X_i, u_i)_{i \in I}$, each player i has a convex and compact strategy space $X_i \subset \mathbb{R}^l$ and a payoff function u_i that associates the *Sharing Rule* defined by $S_i : X \to [0, \overline{s}]$ with $\overline{s} \in (0, +\infty)$. That is to say, each player has a payoff function $u_i : X \to \mathbb{R}$ with the form $u_i(x_i, x_{-i}) = F_i[x_i, S_i(x_i, x_{-i})]$ where $F_i : X_i \times [0, \overline{s}] \to \mathbb{R}$ and u_i is bounded.[13]

[13] For more details on this model, see Rothstein [2007] and Nessah and Tian [2011]

We characterize the existence of pure strategy Nash equilibrium by using Theorem 3.1 as follows.

PROPOSITION 5.1 *The shared resource game* $R = (X_i, u_i)_{i \in I}$ *where* $u_i(x_i, x_{-i}) = F_i[x_i, S_i(x_i, x_{-i})]$ *has a pure strategy Nash equilibrium if and only if there is a nonempty, dominant, compact and convex subset* X^0 *of* X *such that* R *is* $\overline{u}$*-weakly continuous secure on* X^0.

PROOF. See the proof of Theorem 3.1. ■

6 CONCLUSION

We establish a new condition of continuity called *weakly continuous secure*. We offer new Nash equilibrium existence results for a large class of discontinuous games, which rely on weakly continuous secure. Theorem 3.1 characterizes the existence of a pure strategy Nash equilibrium by showing that weakly continuous security on a compact, convex and dominant subset X^0 in X, is necessary while it is sufficient under the boundedness of $G = (X_i, u_i)_{i \in I}$. Proposition 3.2 and Corollary 3.2 give some sufficient condition of the weakly continuous security. These results permit us to significantly weaken the key assumptions, such as continuity and quasiconcavity on the existence of Nash equilibria.

Our results strictly generalize almost the existing theorems on the existence of pure strategy Nash equilibrium such as those in Barelli and Meneghel [2013], McLennan, Monteiro and Tourky [2011], Bich and Laraki [2012], Carmona [2009, 2011], Barelli and Soza [2009], Reny [1999, 2009], Nessah [2011], Prokopovych [2011, 2012]. We also provide some examples illustrating the applicability of our general results, when the existing theorems for pure strategy Nash equilibria fail to hold. It can be easily generalized to the existence of symmetric pure strategy, mixed strategy Nash, and Bayesian Nash equilibria.

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