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Abstract

To analyze the impact of background risks, decreasing absolute risk aversion (DARA) must be combined with other restrictions on the shape of the utility function in order to make preferences risk vulnerable. In this note, we indicate that risk vulnerability can also be associated with the sole assumption of decreasing downside risk aversion (DDRA). That is, no matter how absolute risk aversion changes with wealth, DDRA in the Arrow-Pratt sense and DDRA in the Ross sense are shown to be respectively necessary and sufficient for a background risk to raise the aversion to other independent risks.

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Introduction

The impact of zero mean background risks on the demand for insurance has been first addressed by Doherty and Schlesinger (1983). Since then, the way background risks modify the propensity to make risky decisions has attracted much attention. Among other papers on the topic [1], Gollier and Pratt (1996), Eeckhoudt, Gollier and Schlesinger (1996) and Gollier (2001) established necessary and/or sufficient conditions on the shape of the utility function such that the behavior seen as the most plausible (i.e. zero-mean background risks raise risk aversion, a preference termed "risk vulnerability") is obtained. In these papers, the DARA assumption together with some technical conditions is required to guarantee risk vulnerable preferences.

In the present note, we indicate that risk vulnerability can be associated with the notion of downside risk aversion [2] (DRA). More precisely, to obtain that risk taking falls when the decision maker faces an independent zero mean background risk: 1) decreasing downside risk aversion (DDRA) in the Arrow (1965) and Pratt (1964) sense is necessary and; 2) DDRA in the Ross (1981) sense is sufficient. Our contribution thus relates risk vulnerability to a unique and easily interpretable concept that dispenses us from any assumption about the way the coefficient of absolute risk aversion changes with wealth.

Besides, our work also offers an application to the notion of DRA which is considered as theoretically appealing but not yet associated with a specific economic behavior (Huang (2012)).

Our note is organized as follows. In section 1 we define DDRA as well as the structure of the problem. Section 2 contains the main result of the note and we conclude in section 3.

1 Downside risk aversion

We know from Menezes, Geiss and Tressler (1980) that:

$$R = \frac{1}{2}\left[u(w-k) + E[u(w+\widetilde{x})]\right] - \frac{1}{2}\left[E[u(w-k+\widetilde{x})] + u(w)\right] \gtreqless 0 \text{ if } u''' \gtreqless 0 \quad (1)$$

where w denotes the initial wealth, $\widetilde{x}$ a zero mean risk, and k a potential loss such that $k > 0$. R can be interpreted as the utility premium associated with the misapportionment of risk $\widetilde{x}$ (see Eeckhoudt and Schlesinger (2006)).

Expanding to the second order terms involving $\widetilde{x}$ we have:

$$R \simeq \frac{1}{2}\frac{\sigma^2_{\widetilde{x}}}{2}[u''(w) - u''(w-k)] \quad (2)$$

or, for small k:

$$R \simeq \frac{1}{4}\sigma^2_{\widetilde{x}} k u'''(w) \quad (3)$$

1. See for example Pratt and Zeckhauser (1987), Eeckhoudt and Kimball (1992), Kimball (1993).

2. See Modica and Scarsini (2005), Jindapon and Neilson (2007), Crainich and Eeckhoudt (2008), Li (2009), Denuit and Eeckhoudt (2010) and Wang and Li (2011) for the introduction, the interpretation and the exploitation of this notion.

Returning to Equation (1), R can be made equal to zero under $u''' > 0$ by adding a positive amount of money m to w in $u(w)$. m is thus the amount of money necessary to compensate the downside risk averse individual for the risk misapportionment. Using the technique presented by Crainich and Eeckhoudt (2008), we obtain:

$$m \simeq \frac{1}{4}\sigma_{\tilde{x}}^2 k \frac{u'''(w)}{u'(w)} \tag{4}$$

where $\frac{u'''(w)}{u'(w)}$ is the coefficient of downside risk aversion.

If downside risk aversion is decreasing in wealth, we have:

$$\forall w \quad \frac{\partial}{\partial w}\left(\frac{u'''(w)}{u'(w)}\right) \leq 0 \tag{5}$$

Note that this assumption is equivalent to absolute temperance higher than absolute risk aversion.

If the coefficient of absolute downside risk aversion is decreasing in the Ross sense (Ross-DDRA), we formally have:

$$\forall t \; \forall w \quad \frac{\partial}{\partial w}\left(\frac{u'''(t+w)}{u'(w)}\right) \leq 0 \tag{6}$$

Ross-DDRA is equivalent to absolute temperance higher than absolute risk aversion in the sense of Ross. This condition - along with DARA - is shown to be necessary and sufficient for a second degree stochastic change in background risk to increase risk aversion with respect to foreground risk (see Eeckhoudt, Gollier and Schlesinger (1996)).

We show in the next section how these assumptions determine portfolio choices in the presence of a zero mean independent background risk.

2 The main result

Let us consider a decision-maker who can invest his wealth z either in a safe asset or in a risky one $\widetilde{y}$ with a net return above the safe one ($E(\widetilde{y}) > 0$).

As is well known, the first order condition for a maximum is written:

$$E[\widetilde{y}u'(z+\widetilde{y})] = 0 \tag{2.1}$$

assuming, without loss of generality, that the optimal holding of the risky asset amounts to 1.

Intuition suggests that the introduction of an independent zero mean background risk $\widetilde{x}$ should reduce the demand for the risky asset *i.e.*

$$E[\widetilde{y}u'(z+\widetilde{y}+\widetilde{x})] \leq 0 \tag{2.2}$$

We now show that decreasing downside risk aversion (DDRA) and decreasing downside risk aversion in the Ross sense (Ross-DDRA) are respectively necessary and sufficient to obtain this behaviour.

Formally, we show that when $E(\tilde{x}) = 0$, then:

$$[E(\tilde{y}u'(z+\tilde{y})) = 0 \Rightarrow E(\tilde{y}u'(z+\tilde{y}+\tilde{x})) \leq 0] \;\Rightarrow\; \text{DDRA} \tag{7}$$

and:

$$[E(\tilde{y}u'(z+\tilde{y}))=0 \Rightarrow E(\tilde{y}u'(z+\tilde{y}+\tilde{x}))\leq 0] \;\Leftarrow\; \text{Ross-DDRA} \tag{8}$$

Let us first remind that the decrease in risk taking caused by the introduction of a background risks corresponds to the vulnerability condition set out by Gollier and Pratt (1996).

$$[E(\tilde{y}u'(z+\tilde{y}))=0 \Rightarrow E(\tilde{y}u'(z+\tilde{y}+\tilde{x}))\leq 0] \;\Leftrightarrow\; \frac{E(u''(w+\tilde{x}))}{u''(w)} \geq \frac{E(u'(w+\tilde{x}))}{u'(w)} \tag{9}$$

Another important and new result is as follows:

$$\frac{E(u''(w+\tilde{x}))}{u''(w)} \geq \frac{E(u'(w+\tilde{x}))}{u'(w)} \;\Leftrightarrow\; \frac{u'(w+t)}{u'(w)}-\frac{u''(w+t)}{u''(w)} \leq \left(\frac{u''(w)}{u'(w)}-\frac{u'''(w)}{u''(w)}\right)t \tag{10}$$

So, defining the function ζ as below:

$$\zeta_w(t)=\frac{u'(w+t)}{u'(w)}-\frac{u''(w+t)}{u''(w)} \tag{11}$$

The relationships (9) and (10), which are proved in the appendix, lead to the following equivalence between the attitude towards background risks and a condition on the function ζ:

$$[E(\tilde{y}u'(z+\tilde{y}))=0 \Rightarrow E(\tilde{y}u'(z+\tilde{y}+\tilde{x}))\leq 0] \;\Leftrightarrow\; \zeta_w(t)\leq \zeta_w'(0)\,t \quad \forall t \tag{12}$$

Using this latter equivalence, we first show that DDRA is required to obtain that decisions makers are more risk averse to foreground risks in the presence of background risks.

Proposition 2.1 (Necessary condition)**.**

$$\zeta_w(t)\leq \zeta_w'(0)\,t \quad \forall t \;\Rightarrow\; \mathit{DDRA}$$

Proof. This global property implies the local concavity of $\zeta_w(.)$ at $t=0$, so $\zeta_w''(0)\leq 0$. To see this, recall that by the Taylor-Young formula, we have for t small:

$$\zeta_w(t)=\zeta_w(0)+\zeta_w'(0)\,t+\frac{\zeta_w''(0)\,t^2}{2}+o(t^2)$$

Because $\zeta_w(0)=0$, and the right-hand-side of equivalence (12), we have:

$$\frac{\zeta_w''(0)\,t^2}{2}+o(t^2)\leq 0$$

for t small, so that, necessarily:

$$\zeta_w''(0)\leq 0$$

Differentiating $\zeta_w(.)$ twice, we obtain:

$$\zeta_w''(t)=\frac{u'''(w+t)}{u'(w)}-\frac{u''''(w+t)}{u''(w)}$$

and:

$$\zeta''_w(0) = \frac{u'''(w)}{u'(w)} - \frac{u''''(w)}{u''(w)} \leq 0$$

$\zeta''_w(0) \leq 0$ is therefore equivalent to:

$$u''''(w)u'(w) - u'''(w)u''(w) \leq 0$$

so to:

$$\frac{\partial}{\partial w}\left(\frac{u'''(w)}{u'(w)}\right) \leq 0$$

We thus have:

$$[E(\tilde{y}u'(z+\tilde{y})) = 0 \Rightarrow E(\tilde{y}u'(z+\tilde{y}+\tilde{x})) \leq 0] \Rightarrow \text{DDRA}$$

□

Using again the equivalence (12), we can now demonstrate that decreasing downside risk aversion in the Ross sense is sufficient to make individuals risk vulnerable.

Proposition 2.2 (Sufficient condition)**.**

$$\textit{Ross-DDRA} \;\Rightarrow\; \zeta_w(t) \leq \zeta'_w(0)\; t \quad \forall t$$

Proof. The assumption:

$$\forall t\; \forall w \quad \frac{\partial}{\partial w}\left(\frac{u'''(t+w)}{u'(w)}\right) \leq 0$$

can be written equivalently as:

$$\forall t\; \forall w \quad u''''(t+w)u'(w) - u'''(t+w)u''(w) \leq 0$$

which yields the convexity everywhere of $\zeta_w(.)$:

$$\forall t \quad \zeta''_w(t) = \frac{u'''(w+t)}{u'(w)} - \frac{u''''(w+t)}{u''(w)} \leq 0$$

This convexity implies that $\zeta_w(.)'$ is decreasing, or:

$$\forall t \geq 0 \quad \zeta'_w(t) \leq \zeta'_w(0)$$

and therefore:

$$\forall t \geq 0 \quad \zeta_w(t) = \int_0^t \zeta'_w(s)ds \leq \int_0^t \zeta'_w(0)ds = \zeta'_w(0)t$$

which is can be readily extended to any value of t to yield the desired result.

□

Note finally that the use of the term ‘Ross decreasing downside risk aversion’ or ‘decreasing downside risk aversion in the sense of Ross’ is justified by the following proposition.

Proposition 2.3. *We have the equivalence:*

$$\forall t \ \frac{u'''(t+.)}{u'(.)} \searrow \Leftrightarrow \ \min T \geq \max A \ \Leftrightarrow \ \exists \lambda \mid \forall w \ T(w) \geq \lambda \geq A(w)$$

Proof. Using the results in the previous proof, we readily show that:

$$\forall t \ \forall w \quad \frac{\partial}{\partial w}\left(\frac{u'''(t+w)}{u'(w)}\right) \leq 0$$

is equivalent to:

$$\forall t \ \forall w \quad -\frac{u''''(w+t)}{u'''(w+t)} \geq -\frac{u''(w)}{u'(w)}$$

This simply means that the temperance and aversion coefficients T and A satisfy:

$$\forall x \ \forall y \quad T(y) \geq A(x)$$

This is equivalent to:

$$\forall x \quad \min_y (T(y)) \geq A(x)$$

and then to:

$$\min_y (T(y)) \geq \max_x (A(x))$$

Choosing $\lambda \in [\max_x (A(x)), \min_y (T(y))]$, the remaining equivalence can be immediately obtained. □

Conclusion

Conditions for preferences to exhibit risk vulnerability are summarized in Gollier (2001). These conditions (whether necessary, sufficient or both) are such that DARA is invariably associated with risk vulnerable preferences.

In this note, we indicate that the introduction of a zero mean background risk may raise the aversion to other independent risks even if the utility function is not DARA. We instead demonstrate that DDRA is necessary for risk vulnerability. Besides, DDRA in the Ross sense is also shown to be sufficient to generate the same preference.

Appendix

The diffidence theorem

We recall here the diffidence theorem in the form set out in Gollier (2001). This theorem gives a necessary and sufficient condition on functions f_1 and f_2 satisfying for any random variable $\tilde{x}$ of bounded support $[a, b]$:

$$E\left(f_1(\tilde{x})\right) = 0 \Rightarrow E\left(f_2(\tilde{x})\right) \leq 0 \tag{13}$$

Provided the following three conditions are satisfied:

– $\exists x_0 \mid f_1(x_0) = f_2(x_0) = 0$

– f_1 and f_2 are twice differentiable at x_0

– $f_1'(x_0) \neq 0$

the inequality:

$$\forall x \in [a, b] \quad f_2(x) \leq \frac{f_2'(x_0)}{f_1'(x_0)} f_1(x) \tag{14}$$

is equivalent to (13).

Proof of equivalence (9):

Recall the left-hand-side of equivalence (9):

$$E(\tilde{y}u'(z + \tilde{y})) = 0 \Rightarrow E(\tilde{y}u'(z + \tilde{y} + \tilde{x})) \leq 0 \tag{15}$$

Defining:

$$v(w) = E(u(w + \tilde{x}))$$

we have the equivalent implication:

$$E(\tilde{y}u'(z + \tilde{y})) = 0 \Rightarrow E(\tilde{y}v'(z + \tilde{y})) \leq 0 \tag{16}$$

Using the diffidence theorem, we are led to define:

$$f_1(t) = tu'(z + t)$$

and:

$$f_2(t) = tv'(z + t)$$

Set $t_0 = 0$. The three conditions of the theorem are satisfied, observing that:

$$f_1(0) = 0.u'(z + 0) = 0$$

and:

$$f_2(0) = 0.v'(z + 0) = 0$$

Computing:

$$f_1'(t) = u'(z + t) + tu''(z + t)$$

and:

$$f_2'(t) = v'(z + t) + tv''(z + t)$$

we obtain:

$$f_1'(0) = u'(z)$$

and:

$$f_2'(0) = v'(z)$$

The diffidence theorem yields the following inequality:

$$f_2(t) \leq \frac{f_2'(0)}{f_1'(0)} f_1(t)$$

so:

$$tv'(z+t) \leq \frac{v'(z)}{u'(z)} tu'(z+t) \tag{17}$$

For $t \geq 0$, this inequality is equivalent to:

$$\frac{v'(z+t)}{u'(z+t)} \leq \frac{v'(z)}{u'(z)} \tag{18}$$

and for $t \leq 0$, it is equivalent to:

$$\frac{v'(z+t)}{u'(z+t)} \geq \frac{v'(z)}{u'(z)} \tag{19}$$

so that equation (17) is equivalent to the function:

$$h = \frac{v'}{u'}$$

being decreasing. Then,

$$h' \leq 0$$

is equivalent to:

$$v''u' - v'u'' \leq 0$$

so to:

$$\frac{v''}{v'} \leq \frac{u''}{u'}$$

(bigger risk aversion in the presence of a background risk)
or to, thanks to risk-aversion:

$$\frac{E(u''(w+\tilde{x}))}{u''(w)} \geq \frac{E(u'(w+\tilde{x}))}{u'(w)} \quad \forall w$$

Proof of equivalence (10)

We rewrite the left-hand-side of equivalence (10) as follows:

$$E(\tilde{x}) = 0 \Rightarrow \frac{E(u'(w+\tilde{x}))}{u'(w)} - \frac{E(u''(w+\tilde{x}))}{u''(w)} \leq 0 \tag{20}$$

Using the diffidence theorem, we define:

$$g_1(t) = t$$

and:

$$g_2(t) = \frac{u'(w+t)}{u'(w)} - \frac{u''(w+t)}{u''(w)}$$

We check that:

$$g_1(0) = 0$$

and:

$$g_2(0) = \frac{u'(w+0)}{u'(w)} - \frac{u''(w+0)}{u''(w)} = 1 - 1 = 0$$

The conditions of the theorem are readily satisfied. Compute also:

$$g_1'(t) = 1$$

and:

$$g_2'(t) = \frac{u''(w+t)}{u'(w)} - \frac{u'''(w+t)}{u''(w)}$$

So that:

$$g_1'(0) = 1$$

and:

$$g_2'(0) = \frac{u''(w)}{u'(w)} - \frac{u'''(w)}{u''(w)}$$

The diffidence theorem gives the equivalence between the implication (20) and the following condition:

$$g_2(t) \leq \frac{g_2'(0)}{g_1'(0)} g_1(t)$$

Replacing with the expressions obtained above, we obtain:

$$\frac{u'(w+t)}{u'(w)} - \frac{u''(w+t)}{u''(w)} \leq \left(\frac{u''(w)}{u'(w)} - \frac{u'''(w)}{u''(w)} \right) t$$

which is the required expression.

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