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Modeling the rebalancing slippage of Leveraged Exchange-Traded Funds

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Abstract

Leveraged exchange-traded funds are designed to track a multiple of the daily return of an underlying benchmark index. In order to keep a fixed exposure to the benchmark index, leveraged ETFs have to rebalance their positions everyday, generating a structural 'rebalancing slippage' which has been documented in several empirical studies.

This paper quantifies the rebalancing slippage of leveraged ETFs by developing a tractable model for the dynamics of leveraged funds, which takes into account the impact of active management by leveraged ETFs. We characterize the rebalancing strategy of the leveraged fund and its impact on the value of the leveraged ETF and we model its dynamics in discrete-time. We show that the rebalancing impact systematically diminishes the daily return of the leveraged ETF and that, over a holding period of more than one day, leveraged ETFs develop a tracking-error which can be decomposed between a compounding deviation – that has already been documented and quantified in previous studies – and a rebalancing deviation. The study of the continuous-time limit of the multi-period model allows us to obtain analytical formulas for the rebalancing slippage and the tracking-error of the leveraged ETF. Our theoretical results are consistent with empirical studies which find that tracking-error and rebalancing impact are larger in periods of high volatility and for leveraged ETFs with negative leverage ratios.

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1 Introduction

Leveraged exchange-traded funds (leveraged ETFs) are innovative financial products which are designed to amplify or invert the exposure of investors to a reference benchmark index. The first wave of leveraged ETFs was introduced by ProShares in June 2006: Ultra ProShares provide, every trading day, twice the daily return of the underlying index. Since then, the market for leveraged ETF expanded tremendously; there are now over 250 leveraged ETFs, managing more than $ 30 billion of assets. Nowadays, there exist leveraged ETFs with leverage ratio equal to 2 or 3, – bull leveraged ETFs – which are designed to deliver, on a daily basis, twice or three times the underlying index daily return and leveraged ETFs with leverage ratio equal to -1, -2 or -3, – bear leveraged ETFs – which deliver respectively the inverse, twice the inverse or three times the inverse of the benchmark's daily return. The sharp increase in the number of existing leveraged ETFs and the size of their assets under management, documented in [Tang and Xu, 2011, TableA2], reflects the growing popularity of such financial innovation which was considered as an easy and inexpensive way to build leveraged positions without using options or margin.

While leveraged ETFs deliver multiples of a benchmark's return on a *daily basis*, the losses incurred by long-term investment in leveraged ETFs have raised growing concerns for regulators ([Haryanto et al., 2012, Section 1]). Several empirical studies ([Lu et al., 2009], [Avellaneda and Zhang, 2010], [Tang and Xu, 2011]) show that over holding periods of one quarter to one year, leveraged ETFs deviate significantly from their underlying benchmark. [Tang and Xu, 2011] show empirically that this tracking-error is due to two major effects: a *compounding effect* and a *rebalancing effect*, while taxes and management fees appear to be negligible.

Theoretical studies such as [Cheng and Madhavan, 2009], [Lu et al., 2009] and [Avellaneda and Zhang, 2010] explain quantitatively the compounding effect. They model the price dynamics of a leveraged ETF in discrete-time and continuous-time and show that the fact that the leveraged fund keeps a fixed exposure to the underlying index mechanically results in a (computable) compounding deviation over the long term.

As far as the rebalancing effect is concerned, [Tang and Xu, 2011] show that it is at least as significant as the compounding effect. [Avellaneda and Dobi, 2012] also estimate this effect from empirical data and explain this rebalancing deviation by leveraged funds' active management. As described in [Haryanto et al., 2012], in order to deliver levered returns, leveraged ETFs use total return swaps with counterparties which have to hedge their exposure to the underlying benchmark index. The hedging strategy implies buying high and selling low, because when the benchmark index goes up (resp. down), bull leveraged ETFs have to increase (resp. decrease) their exposure to the index and hence buy (resp. sell) while bear leveraged ETFs have to decrease (resp. increase) their exposure and also buy back their short position (resp. sell further). Note that such trend-following hedging strategy generated by keeping a fixed leverage ratio has already been described in a more general context of leverage for financial institutions ([Danielsson and Shin, 2003], [Adrian and Shin, 2008], [Shin, 2010], [Greenwood and Thesmar, 2011]). As leveraged funds target daily returns, this rebalancing is typically executed near the market close and anticipated by the market, generating a significant impact for the leveraged fund.

In this paper, we develop a quantitative framework which models the dynamics of leveraged ETFs in the presence of rebalancing costs. In Section 2, we model the rebalancing strategy followed by the leveraged fund and its impact on the leveraged ETF in discrete time and we deduce the dynamics of the leveraged ETF in the presence of rebalancing impact (Section 2.1). In the case of a linear impact, we give the explicit dynamics of the leveraged ETF and show how its return over several days deviates from its benchmark. We give an explicit formula for the tracking-error, which can be decomposed between a compounding effect and a rebalancing effect (Sections 2.2 and 2.3). In Section 3, we study the continuous-time limit of the multi-period model (Section 3.1), which allows us to obtain tractable formulas for the rebalancing slippage and to disentangle the contribution of the rebalancing slippage and the compounding effect to the tracking-error of leveraged ETFs (Section 3.2).

2 Impact of active management by leveraged funds

2.1 Multi-period model

Consider a financial market where trading is possible at discrete dates $t_k = k\Delta t$, where Δt is taken to be a trading day. The market comprises an underlying benchmark index and a leveraged ETF with leverage ratio β. Denote respectively S_k and L_k the value of the benchmark and leveraged ETF at date t_k.

The underlying benchmark index At each period, the value of the benchmark moves exogenously:

$$S_{k+1} = S_k \exp\left((\mu - \frac{v_k}{2})\Delta t + \sqrt{\Delta t}\sqrt{|v_k|}\xi_{k+1}\right) \tag{1}$$

where μ is the drift of the benchmark and v is the variance process, which follows a CIR process:

$$v_{k+1} = v_k + \kappa(\lambda - v_k)\Delta t + \theta\sqrt{|v_k|}\sqrt{\Delta t}\nu_{k+1} \tag{2}$$

Here, $\lambda \geq 0$ is the long term variance, $\kappa \geq 0$ is a coefficient of mean-reversion, $\theta \geq 0$ is a term of volatility of volatility. $(\xi_k)_{k\geq 0}$ and $(\nu_k)_{\geq 0}$ are sequences of iid centered random variables with variance equal to one and correlation equal to $\mathbb{E}(\xi_k \nu_k) = \rho$.

The leveraged ETF We consider a leveraged ETF with leverage ratio β, which means that, every day, the return of the leveraged ETF should be equal to β times the return of the underlying index. Note that we do not take into account management fees and taxes which are negligible compared to the rebalancing slippage of leveraged ETF ([Avellaneda and Dobi, 2012]).

In order to achieve the required exposure, the leveraged fund replicates the leveraged ETF by building a portfolio made of underlying index (δ_k shares owned at date t_k) and cash (rate r). The leveraged ETF can be decomposed as follows:

$$L_k = \underbrace{\delta_k S_k}_{part\ in\ underlying\ ETF} + \underbrace{L_k - \delta_k S_k}_{part\ in\ cash}$$

Notice that, in practice, as discussed in Section 1, the leveraged fund does not build this replicating porftolio itself but rather engages in a total return swap and the counterparty of the swap generally builds this replicating/hedging porftolio. The effect is the same on the leveraged ETF and we consider, for clarity purpose only, that the hedging portfolio is built by the leveraged fund.

Typically, bull leveraged ETFs ($\beta > 0$) borrow money ($L_k - \delta_k S_k \leq 0$) and buy the benchmark ($\delta_k \geq 0$) while bear leveraged ETFs ($\beta < 0$) sell short the benchmark ($\delta_k \leq 0$ and $L_k - \delta_k S_k \geq 0$).

In order to maintain a leverage ratio of β, the leveraged fund has to keep a constant proportion β of its wealth invested in the underlying index, in a self-financing way:

$$\delta_k S_k = \beta L_k \tag{3}$$

The rebalancing impact As discussed in Section 1, due to the change in underlying index value, given in (1) and (2), and in order to maintain a constant exposure of β, the leveraged fund has to rebalance its positions in underlying index from δ_k to δ_{k+1}. Typically, if the underlying benchmark has a positive (resp. negative) return between t_k and t_{k+1}, the leveraged fund has to buy (resp. sell) the benchmark.

As documented in several studies such as [Tang and Xu, 2011], such active management diminishes the return of the leveraged ETF. This impact of rebalancing is originated by the fees paid by the leveraged fund for its rebalancing trade and, most importantly, by the fact that, at the end of the day, the rebalancing trade from the leveraged fund is expected by other financial institutions who temporarily adjust their bid-ask spread on the underlying index, generating a loss for the leveraged fund (see Remark 2.3): for example, if the underlying benchmark increased during the day, the leveraged fund has to buy the benchmark and market-makers will tend to shift their bid-ask spread on the index towards higher levels, until the rebalancing trade is operated.

The active management hence generates costs for the leveraged fund and we model the loss due to rebalancing in a non parcimonious way, as a function of the size of the rebalancing trade ($S_{k+1}(\delta_{k+1} - \delta_k)$). The rebalancing impact is given by:

$$\Phi(S_{k+1}(\delta_{k+1} - \delta_k)) \tag{4}$$

Φ is an impact/cost function, taken to be non-negative (the fund always loses money due to the impact of its rebalancing trade) and verifying $\Phi(0) = 0$ (no rebalancing impact if there is no rebalancing).

Dynamics of the leveraged ETF The price dynamics of the leveraged ETF should then verify

$$L_{k+1} - L_k = \delta_k(S_{k+1} - S_k) + (L_k - \delta_k S_k) r\Delta t - \underbrace{\Phi(S_{k+1}(\delta_{k+1} - \delta_k))}_{rebalancing\ impact} \quad (5)$$

Note that, in (5), the left-hand side depends on δ_{k+1} because $L_{k+1} = \frac{\delta_{k+1} S_{k+1}}{\beta}$ and so does the right-hand side, through the term of rebalancing impact. As a consequence, this equation is not verified for all impact functions Φ. We now exhibit an assumption under which (5) is verified.

Assumption 2.1 *There exists $K \geq 0$ such that*

$$K < \frac{1}{|\beta|} \qquad and \qquad \forall x \in \mathbb{R}, \qquad \Phi(x) \leq K|x|$$

Proposition 2.2 *Under Assumption 2.1, $(S_k, v_k, L_k)_{k\geq 0}$ defined by* (1), (2), (3) *and* (5) *is a Markov chain. In particular, there exists a unique self-financing rebalancing strategy $(\delta_k)_{k\geq 0}$ for the leveraged fund which allows to maintain a fixed exposure of β.*

Proof Assume that we know the value of (S_k, v_k, L_k) and (ξ_{k+1}, ν_{k+1}), which are random variables that are independant from the past. (1) and (2) directly give the value of S_{k+1} and v_{k+1}. In particular, we remark that (1) defines a strictly positive price process for the underlying index. Using (3), we can write (5) as follows:

$$\frac{\delta_{k+1} S_{k+1} - \delta_k S_k}{\beta} = \delta_k(S_{k+1} - S_k) + (L_k - \delta_k S_k) r\Delta t - \Phi(S_{k+1}(\delta_{k+1} - \delta_k))$$

which means that

$$(\delta_{k+1} - \delta_k) S_{k+1} + \beta\Phi(S_{k+1}(\delta_{k+1} - \delta_k)) = \delta_k(S_k - S_{k+1}) + \beta\delta_k(S_{k+1} - S_k(1 + r\Delta t)) + L_k \beta r \Delta t$$

Under Assumption 2.1, the fixed point theorem states that there exists a unique value for $(\delta_{k+1} - \delta_k) S_{k+1}$ such that the previous equation is verified. As a consequence, given $\delta_k = \frac{\beta L_k}{S_k}$ and $S_{k+1} > 0$ by (1), there exists a unique value for δ_{k+1} and hence for L_{k+1}. This proves that $(S_k, v_k, L_k)_{k\geq 0}$ is a Markov chain.

(5) shows that, every day, the return of the leveraged ETF is diminished by the rebalancing costs. This effect is distinct from the 'compounding effect' described in [Avellaneda and Zhang, 2010] and is due to the active management by the leveraged fund, as documented by [Tang and Xu, 2011] and [Avellaneda and Dobi, 2012].

As a first order approximation, one can consider that the impact function is linear, as done in [Almgren and Chriss, 2000, Almgren and Lorenz, 2006]: $\Phi(x) = c|x|$. If the fund makes a trade of size x (in \$), it loses $c|x|$. Typically, c can span between 1 to 500 basis points, which makes Assumption 2.1 very reasonable.

In the presence of rebalancing impact, not only the value of the leveraged ETF is systematically diminished, but also the rebalancing strategy δ for the fund is modified. When there are no costs, the 'fundamental' rebalancing strategy $\tilde{\delta}$ can be deduced from (5) with $\Phi = 0$, and is given by:

$$\tilde{\delta}_{k+1} = \tilde{\delta}_k + (\beta - 1)\tilde{\delta}_k \left(1 - \frac{(1 + r\Delta t)S_k}{S_{k+1}}\right) \tag{6}$$

A leveraged fund which does not take into account its rebalancing costs when determining its end-of-day rebalancing trade and follows the rebalancing strategy $\tilde{\delta}$ will fail to keep an exposure of β to the underlying index. Proposition 2.2 shows that for a given impact function Φ verifying Assumption 2.1, there exists a unique rebalancing strategy δ which allows to keep a fixed leverage ratio of β. This strategy can be determined by solving (numerically) (5) and is different from the 'naive' rebalancing strategy $\tilde{\delta}$.

Remark 2.3 *As suggested by empirical studies such as [Haryanto et al., 2012] which point out the fact that the size of the leveraged ETF market is small compared to the size of the ETF market, we consider that the active management of leveraged ETFs does not impact the underlying index in a permanent way but only temporarily (see also [Greenberg, 2011]). See [Almgren and Chriss, 2000] for a discussion on temporary and permanent price impacts.*

2.2 Case of linear rebalancing impact

In this section, we study the case of a linear rebalancing impact: $\Phi(x) = c|x|$, with $c < \frac{1}{|\beta|}$. In this situation, it is possible to compute explicitely the rebalancing strategy by the leveraged fund and the price dynamics of the leveraged ETF. For clarity purpose, we also assume here that $r = 0$.

(3) and (5) can be written as

$$\frac{\delta_{k+1}S_{k+1} - \delta_k S_k}{\beta} = \delta_k(S_{k+1} - S_k) - c|\delta_{k+1} - \delta_k|S_{k+1}$$

As a consequence:

- when the underlying index has a positive return during period k ($S_{k+1} \geq S_k$):

$$\delta_{k+1} = \delta_k + \frac{\beta - 1}{1 + c\beta}\delta_k \left(1 - \frac{S_k}{S_{k+1}}\right)$$

$$L_{k+1} - L_k = \delta_k(S_{k+1} - S_k) - c\frac{|(\beta - 1)\delta_k(S_{k+1} - S_k)|}{1 + \beta c}$$

- when the underlying index has a negative return during period k ($S_{k+1} \leq S_k$):

$$\delta_{k+1} = \delta_k + \frac{\beta - 1}{1 - c\beta}\delta_k \left(1 - \frac{S_k}{S_{k+1}}\right)$$

$$L_{k+1} - L_k = \delta_k(S_{k+1} - S_k) - c\frac{|(\beta - 1)\delta_k(S_{k+1} - S_k)|}{1 - \beta c}$$

Keeping a constant exposure to the underlying index implies a 'trend-following' rebalancing strategy (buy high and sell low)

$$S_{k+1} \geq S_k \Leftrightarrow \delta_{k+1} \geq \delta_k$$

Furthermore, the presence of costs systematically decreases the volume of purchases after an increase in ETF value and increases the number of sales after a decrease in ETF value. Not surpringly, when there are no rebalancing costs ($c = 0$), we find the 'fundamental' rebalancing strategy given in (6).

At order one in costs c, the dynamics of the leveraged ETF can be expressed as follows:

$$\underbrace{\frac{L_{k+1} - L_k}{L_k}}_{leveraged\ ETF\ return} = \underbrace{\beta \frac{S_{k+1} - S_k}{S_k}}_{benchmark\ return} \underbrace{- c|\beta(\beta - 1)| \left| \frac{S_{k+1} - S_k}{S_k} \right|}_{rebalancing\ costs} + o(c) \tag{7}$$

(7) shows that, *every day*, the return of the leveraged ETF is systematically reduced by the impact of active management, whereas there is no coumpounding effect over a period of one day. This confirms the empirical results of [Avellaneda and Dobi, 2012], who find that, due to rebalancing costs, leveraged ETFs fail to perform as expected even over one-day holding periods. We see that the more the benchmark moved, the more the leveraged fund has to rebalance and the more it loses in costs. In addition, the term $|\beta(\beta - 1)|$ shows that rebalancing costs are more pronounced for funds with negative leverage.

Let us now study the return of the leveraged ETF over more than one day and distinguish between the compounding effect and the rebalancing effect. It is possible to compute such return over a given number of days. For clarity purpose, let us focus on a two-day return, which contains all the information on the decomposition of the tracking-error between compounding deviation and rebalancing deviation. Note that in Section 3, the study of the continuous-time limit allows us to obtain simple tractable expressions for the tracking-errors over any period of time. By direct computation of (7), we find that:

$$\underbrace{\frac{L_{k+2} - L_k}{L_k}}_{2\ days\ leveraged\ ETF\ return} = \underbrace{\beta \frac{S_{k+2} - S_k}{S_k}}_{benchmark\ return} \underbrace{- \beta(\beta - 1) \frac{S_{k+2} - S_{k+1}}{S_{k+1}} \frac{S_{k+1} - S_k}{S_k}}_{compounding\ effect} \tag{8}$$

$$\underbrace{- c|\beta(\beta - 1)| \left(\left| \frac{S_{k+1} - S_k}{S_k} \right| (1 + \beta \frac{S_{k+2} - S_{k+1}}{S_{k+1}}) + \left| \frac{S_{k+2} - S_{k+1}}{S_{k+1}} \right| (1 + \beta \frac{S_{k+1} - S_k}{S_k}) \right)}_{cumulative\ rebalancing\ costs} + o(c)$$

(8) shows that the tracking-error of the leveraged ETF can be decomposed into a compounding effect deviation and a rebalancing effect deviation. The term of compounding effect has been described in [Cheng and Madhavan, 2009], [Lu et al., 2009] and [Avellaneda and Zhang, 2010]. It is path-dependant and exacerbated for negative leverage ratios. In addition, in our model, we are able to quantify the rebalancing effect, which

was documented in empirical studies ([Tang and Xu, 2011], [Avellaneda and Dobi, 2012]). We see that it is also path-dependant and larger for negative leverage ratios. More interestingly, we see that it is of order one in $\frac{\Delta S}{S}$, whereas the term of compounding effect is of order 2 in $\frac{\Delta S}{S}$, confirming the empirical results in [Tang and Xu, 2011].

2.3 Numerical experiments

In this section, we simulate the multi-period model described previously and study numerically the rebalancing impact of the leveraged ETF. We focus on the case of a linear rebalancing impact function $\Phi(x) = c|x|$, which was studied in detail in Section 2.2 and use the following parameters:

- the leverage ratio is $\beta = 2$
- $S_0 = L_0 = \$100$, which implies that $\delta_0 = 2$
- the stochastic volatility oscillates around 30%: $v_0 = \lambda = 0.09$, $\kappa = 2$ and $\theta = 0.1$
- ξ and ν are standard normal random variables which are independent: $\rho = 0$.

Figure 1 displays, for one scenario, the dynamics of the leveraged ETF L, the rebalancing strategy δ and the underlying index S over a ten-day period, when the cost rate c is equal to 100 basis points (lines in blue). We use, as a benchmark for comparison, the 'fundamental' dynamics of L and of δ in the absence of rebalancing costs (in green), which corresponds to the case $c = 0$. We see that the performance of the leveraged ETF is lowered and the rebalancing strategy, although still 'trend-following' (you have to buy if and only if the value of the underlying index increased), is modified due to the rebalancing impact.

Table 1 gives the average rebalancing slippage (over 10^5 simulations) for different values of c and time horizons. We see that the longer the holding period, the greater the rebalancing slippage. Similarly and not suprisingly, the larger the rebalancing cost, the greater the rebalancing slippage. Interestingly, we find numerically that this slippage is a linear function of the cost rate c, which justifies the approximation at order one in c made in Section 2.2. For $c = 100bp$, which is a realistic value for the cost rate, we see that over a holding period of more than one quarter, the deviation due to the active management by the leveraged fund can be significant.

3 Quantifying the rebalancing slippage of leveraged ETFs

3.1 Continuous-time limit of the multi-period model

In this section, we study the continuous-time limit of the multi-period model, which gives us a tractable formula for the impact of the active management by the leveraged fund on the leveraged ETF's returns. In order to do so, we make the following assumption.

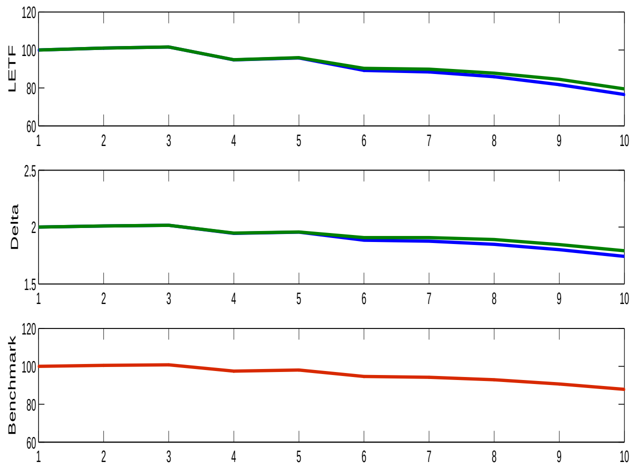

Figure 1: Leveraged ETF and δ dynamics with (line in blue) and without (line in green) rebalancing costs and underlying index dynamics

Rebalancing slippage			
holding period	$c = 1$bp	$c = 10$bp	$c = 100$bp
1 day	$4.2.10^{-6}$	$4.2.10^{-5}$	$4.3.10^{-4}$
1 month	$6.2.10^{-5}$	$6.2.10^{-4}$	$6.2.10^{-3}$
1 quarter	$1.7.10^{-4}$	$1.7.10^{-3}$	$1.7.10^{-2}$
1 year	$7.4.10^{-4}$	$7.3.10^{-3}$	$7.1.10^{-2}$

Table 1: Average rebalancing slippage for different horizons and cost rates

Assumption 3.1 *$\Phi \in \mathcal{C}^2$ and $\mathbb{E}(\nu^2) < \infty$ and there exists $\eta > 0$ such that*

$$\mathbb{E}(\exp(\eta\xi)) < \infty \quad \textit{and} \quad \mathbb{E}(\xi^{\eta+4}) < \infty$$

Theorem 3.2 *Under Assumptions 2.1 and 3.1, the process $\left(S_{\lfloor \frac{t}{\Delta t} \rfloor}, v_{\lfloor \frac{t}{\Delta t} \rfloor}, L_{\lfloor \frac{t}{\Delta t} \rfloor}\right)_{t\geq 0}$ converges weakly on the Skorokhod space $D([0,\infty[,\mathbb{R}^3)$ as Δt goes to 0 to a diffusion process $(S_t, v_t, L_t)_{t\geq 0}$ solution of the stochastic differential equation:*

$$\frac{dS_t}{S_t} = \sqrt{v_t}dW_t + \mu dt$$

$$dv_t = \theta\sqrt{v_t}dB_t + \kappa(\lambda - v_t)dt$$

$$\frac{dL_t}{L_t} = \beta\frac{dS_t}{S_t} + (1-\beta)rdt - \frac{\Phi''(0)}{2}(\beta - 1)^2\beta^2 v_t L_t dt$$

where $(W_t)_{t\geq 0}$ and $(B_t)_{t\geq 0}$ are two Brownian Motions, with correlation ρ.

Proof We know that under Assumption 2.1, the mapping $Id + \beta\Phi$ is invertible. Let us denote its inverse by $\Psi = (Id + \beta\Phi)^{-1}$. Under Assumption 3.1, $\Phi \in \mathcal{C}^2$ and hence $\Psi \in \mathcal{C}^2$. We saw that (5) implies that:

$$(\delta_{k+1}-\delta_k)S_{k+1}+\beta\Phi(S_{k+1}(\delta_{k+1}-\delta_k)) = \delta_k(S_k-S_{k+1})+\beta\delta_k(S_{k+1}-S_k(1+r\Delta t))+L_k\beta r\Delta t$$

which means that:

$$(\delta_{k+1} - \delta_k)S_{k+1} = \Psi\left(\delta_k(S_k - S_{k+1}) + \beta\delta_k(S_{k+1} - S_k(1 + r\Delta t)) + L_k\beta r\Delta t\right)$$

As a consequence, we can write (5) as follows:

$$\begin{aligned} L_{k+1} - L_k &= \delta_k(S_{k+1} - S_k) + (L_k - \delta_k S_k)r\Delta t \\ &-\Phi\left(\Psi\left(\delta_k(S_k - S_{k+1}) + \beta\delta_k(S_{k+1} - S_k(1 + r\Delta t)) + L_k\beta r\Delta t\right)\right) \end{aligned}$$

Let us expand the last term (of impact):

$$\begin{aligned} &\Phi\left(\Psi\left(\delta_k(S_k - S_{k+1}) + \beta\delta_k(S_{k+1} - S_k(1 + r\Delta t)) + L_k\beta r\Delta t\right)\right) \\ &= [\delta_k(S_k - S_{k+1}) + \beta\delta_k(S_{k+1} - S_k(1 + r\Delta t)) + L_k\beta r\Delta t]\,(\Phi o\Psi)'(0) \\ &+\frac{1}{2}[\delta_k(S_k - S_{k+1}) + \beta\delta_k(S_{k+1} - S_k(1 + r\Delta t)) + L_k\beta r\Delta t]^2\,(\Phi o\Psi)''(0) + o(\Delta t) \end{aligned}$$

where $\mathbb{E}(\frac{o(\Delta t)}{\Delta t}) \to 0$ when $\Delta t \to 0$ and (S_k, v_k, L_k) remain in a compact set of $\mathbb{R}^3$. By definition, we have $\Psi(x) = x - \beta\Phi(\Psi(x))$. This implies that $\Psi(0) = 0$, because Ψ is invertible, and $\Psi'(0) = 1$. Note also that under assumption 3.1 and because $\Phi \geq 0$, we have $\Phi'(0) = 0$. As a consequence, $(\Phi \circ \Psi)'(0) = 0$ and $(\Phi \circ \Psi)''(0) = \Phi''(0)$. And so we find that:

$$\begin{aligned} &\Phi\left(\Psi\left(\delta_k(S_k - S_{k+1}) + \beta\delta_k(S_{k+1} - S_k(1 + r\Delta t)) + L_k\beta r\Delta t\right)\right) \\ &= \frac{\Phi''(0)}{2}[\delta_k(S_k - S_{k+1}) + \beta\delta_k(S_{k+1} - S_k(1 + r\Delta t)) + L_k\beta r\Delta t]^2 + o(\Delta t) \\ &= \frac{\Phi''(0)}{2}[(\beta - 1)\delta_k(S_{k+1} - S_k)]^2 + o(\Delta t) \end{aligned}$$

As a consequence, we can write:

$$L_{k+1} - L_k = \delta_k(S_{k+1} - S_k) + (L_k - \delta_k S_k)r\Delta t - \frac{\Phi''(0)}{2}[(\beta - 1)\delta_k(S_{k+1} - S_k)]^2 + o(\Delta t)$$

where $\mathbb{E}(\frac{o(\Delta t)}{\Delta t}) \to 0$ when $\Delta t \to 0$ and (S_k, v_k, L_k) remain in a compact set of $\mathbb{R}^3$. Using the previous result and the exact same methodology as done in the proof of [Cont and Wagalath, 2013b, Theorem 4.1] and [Cont and Wagalath, 2013a, Theorem 2.4], we conclude the proof of Theorem 3.2.

Note that the stochastic differential equation verified by the variance process

$$dv_t = \theta\sqrt{v_t}dB_t + \kappa(\lambda - v_t)dt$$

has a unique strong solution, which is non-negative ([Mao, 1997, Section 9.2]). In the continuous-time limit, the underlying index price process $(S_t)_{t\geq 0}$ follows a classical Heston model ([Heston, 1993]), with stochastic volatility process $(\sqrt{v_t})_{t\geq 0}$. In the next section, we study the dynamics of the leveraged ETF price process, $(L_t)_{t\geq 0}$.

3.2 Analytical formula for the rebalancing slippage

Theorem 3.2 shows that, due to the active management by the leveraged fund, the leveraged ETF returns are diminished in a systematic way. The instantaneous rebalancing slippage is equal to:

$$\frac{\Phi''(0)}{2}(\beta-1)^2\beta^2 v_t L_t$$

It is liquidity-dependent (through the term $\Phi''(0)$) and is larger for negative leverage ratios (the term $(\beta-1)^2\beta^2$ is larger for $\beta=-3$, -2 or -1 than $\beta=2$ or 3). In addition, the larger the leveraged fund (i.e. the greater L_t), the larger its rebalancing impact. Similarly, the larger the underlying index's variance v_t, the larger the rebalancing slippage, which is consistent with the findings of [Tang and Xu, 2011] who show empirically that the rebalancing slippage of leveraged ETFs is more pronounced during volatile periods. Indeed, when there is no rebalancing impact ($\Phi=0$), this rebalancing slippage is equal to zero and the active management by the leveraged fund does not impact the value of the leveraged ETF.

Given the dynamics of L in Theorem 3.2, the leveraged ETF's value can be written as follows:

$$L_t = L_0\left(\frac{S_t}{S_0}\right)^{\beta}\exp\left[(-(\beta-1)rt) - \underbrace{\left(\frac{(\beta-1)\beta}{2}\int_0^t v_s\,\mathrm{d}s\right)}_{compounding\ effect} - \underbrace{\left(\frac{\Phi''(0)}{2}(\beta-1)^2\beta^2\int_0^t v_s L_s\,\mathrm{d}s\right)}_{rebalancing\ effect}\right] \tag{9}$$

Equation 9 shows that the tracking-error of the leveraged ETF over a period $[0,t]$ can be decomposed between:

- a term of compounding deviation, which was already explicited in [Lu et al., 2009] and [Avellaneda and Zhang, 2010]; and
- a term of rebalancing deviation that we express analytically as a function of the model parameters

Both effects add up to create a tracking-error between the leveraged ETF and its benchmark S_t^{β}. Note that this tracking-error will be more pronounced in periods of high volatility.

It is possible to capture this rebalancing slippage of leveraged ETFs by building strategies involving leveraged ETFs and their underlying index, as described in [Jiang and Peterburgsky, 2013]. Consider now that $\beta>0$ and denote L^+ (resp. L^-) the bull (resp. bear) leveraged ETF, with leverage ratio β (resp. $-\beta$). Consider now the three following strategies, for which we can compute the (instantaneous) returns thanks to Theorem 3.2:

- Strategy 1: short L^+, short L^-, with return equal to

$$\frac{\Phi''(0)}{2}\beta^2 v_t\left[(\beta-1)^2L_t^+ + (\beta+1)^2L_t^-\right] - 2r$$

- Strategy 2: short L^+, long β underlying index, with return equal to

$$\frac{\Phi''(0)}{2}(\beta-1)^2\beta^2 v_t L_t^+ + (\beta-1)r$$

- Strategy 3: short L^-, short β underlying index, with return equal to

$$\frac{\Phi''(0)}{2}(\beta+1)^2\beta^2 v_t L_t^- - (\beta+1)r$$

Those three strategies, based on shorting the leveraged funds, benefit from the rebalancing slippage of the leveraged ETFs. Let us illustrate the performance of those strategies in the following cases:

- underlying benchmark: EEM (emerging markets), bull leveraged ETF: EDC (leverage 3), bear leveraged ETF: EDZ (leverage -3)
- underlying benchmark: IYF (real estate), bull leveraged ETF: UYG (leverage 2), bear leveraged ETF: SKF (leverage -2)
- underlying benchmark: RGUSFL (Russell 1000 financial services index), bull leveraged ETF: FAS (leverage 3), bear leveraged ETF: FAZ (leverage -3)

Figure 2 shows an exemple of the performance of strategy 1, while Figures 3, 4 and 5 display the performance of strategies 2 and 3. We see that the rebalancing slippage of leveraged ETFs generates a significant positive trend for those strategies. However, in practice, the cost of shorting leveraged ETFs, that we did not take into account in those Figures, generally compensates the gain in slippage ([Avellaneda and Dobi, 2012]) and reduces significantly the performance of those strategies.

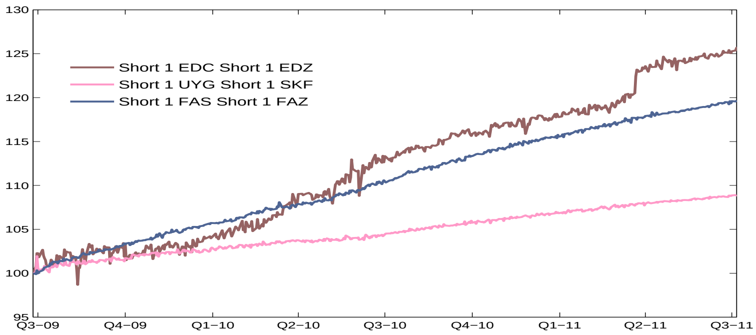

Figure 2: Short leveraged ETF bull and Short leveraged ETF bear

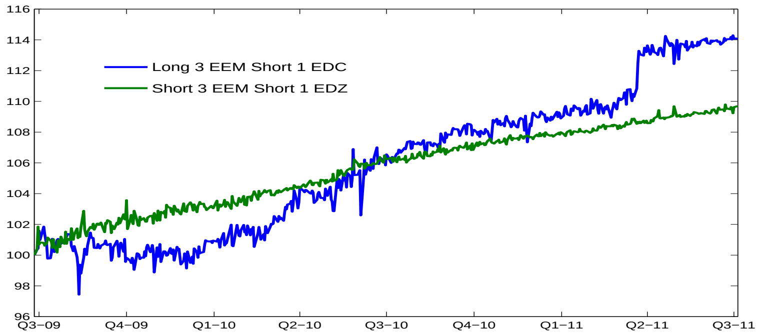

Figure 3: short EDC and long 3 EEM; short EDZ and short 3 EEM

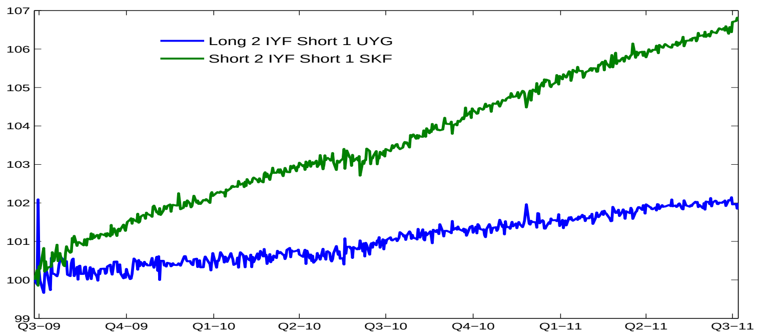

Figure 4: short UYG and long 2 IYF; short SKF and short 2 IYF

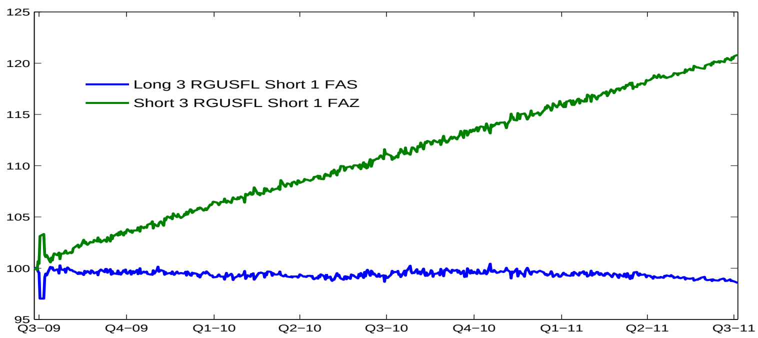

Figure 5: short FAS and long 3 RGUSFL; short FAZ and short 3 RGUSFL

4 Conclusion

The management of leveraged ETFs, which are designed to track the multiple of an underlying benchmark's return on a daily basis, is based on the daily rebalancing of the fund's positions, in order to keep a contant exposure to the benchmark. This active management generates costs which reduce the performance of the leveraged fund in a systematic way ([Tang and Xu, 2011], [Avellaneda and Dobi, 2012]).

In this paper, we develop a quantitative model for the price dynamics of the leveraged ETF, which takes into account the rebalancing impact of the leveraged fund. We first model this dynamics in discrete-time and show that the impact of rebalancing decreases the return of the leveraged ETF even over a one-day holding period. The study of the continuous-time limit of the multi-period model allows us to obtain analytical formulas for the rebalancing slippage of leveraged ETFs. We show that this rebalancing slippage is liquidity-dependent and exacerbated during periods of high volatility and for funds with negative leverage ratios. Our theoretical results allow to disentangle the contribution of the 'rebalancing deviation' and the 'compounding deviation' to the tracking-error of leveraged ETFs and hence model, in a more complete way, the dynamics of leveraged ETFs.

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