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Strategic fire-sales and price-mediated contagion in the banking system

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Abstract

We consider a price-mediated contagion framework in which each bank, after an exogenous shock, may have to sell assets in order to comply with regulatory constraints. Interaction between banks takes place only through price impact. We characterize the equilibrium of the strategic deleveraging problem and we calibrate our model to publicly-available data, the US banks that were part of the 2015 regulatory stress-tests. We then consider a more sophisticated model in which each bank is exposed to two risky assets (marketable and not marketable) and is only able to sell the marketable asset. We calibrate our model using the six banks with significant trading operations and we show that, depending on the price impact, the contagion of failures may be significant. Our results may be used to refine current stress testing frameworks by incorporating potential contagion mechanisms between banks.

Keywords: Fire sales, price-mediated contagion, Nash equilibrium with strategic complementarities, CCAR 2015, macro-prudential stress-tests

1 Introduction

Past financial crises have repeatedly shed light on the critical role played by financial institutions in propagating and amplifying an exogenous adverse shock [Brunnermeier, 2009, Krishnamurthy, 2010, Glasserman and Young, 2016]. This was recently illustrated during the 2007 subprime crisis when a shock in a relatively small asset class, the US subprime mortgages, resulted in magnified losses for numerous financial institutions due to contagion effects. A salient feature of the 2007 crisis is the role played by indirect, rather than direct, contagion effects [Clerc et al., 2016].

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Direct contagion is the result of contractual links between financial institutions, typically debt or OTC derivatives: the failure of a given institution will trigger losses for its counterparties, potentially causing the defaults of other institutions, which will in turn trigger losses for their own counterparties and further failures etc... This direct contagion, generated by counterparty risk, has long been acknowledged as an important source of financial instability and has been studied by academics through network models ([Eisenberg and Noe, 2001, Elsinger et al., 2006, Fouque and Langsam, 2013, Upper and Worms, 2004]). Regulators have recently (partly) tackled counterparty risk by introducing collateral requirements and limitations of large exposures for OTC derivatives trades, see for instance [Glasserman and Young, 2015].

Indirect (or price-mediated) contagion is in some sense a more subtle form of contagion as it occurs through price effects, even in the absence of direct contractual links between institutions: a given financial institution may be forced to sell some assets, pushing prices down and generating losses for all institutions holding the same assets. Such forced sales are generally referred to as *fire sales* and typically occur at a dislocated price when a distressed institution is willing to promptly liquidate part of its portfolio [Diamond and Rajan, 2011, Shleifer and Vishny, 2011]. Depending on the type of financial institution (i.e., depository institution, mutual fund, pension fund, insurance company, hedge fund ...), various reasons may be found to explain these fire sales, for instance collateralized short term financing [Shleifer and Vishny, 2011]. In the case of (insured) depository institutions, everything else equal, fire sales may indeed be triggered by regulatory capital requirements themselves.

One of the common features of the successive Basel regulations is that supervisors consider the following capital ratio to assess the solvency of banks:

$$\text{Risk based capital ratio} := \frac{\text{Total capital}}{\text{Risk Weighed Assets}} \tag{1}$$

Equation (1) is called a risk-based capital ratio (RBC) because its denominator is the total risk-weighted assets (RWA), defined as the sum of a few risk-related RWAs (typically credit, market and operational risks) rather than the total value of assets. In Basel III, the numerator of the RBC is defined as the sum of two types of capital, Tier 1 and Tier 2, where Tier 1 is designed to absorb losses without affecting the business as usual while Tier 2 is designed to absorb losses in case of liquidation of the bank[1]. Basel III regulation imposes that the RBC of a given bank must be *at least 8% at all times*. This means that when the assets of a bank are hit by an adverse shock and the bank's RBC drops below 8%, because it is generally too costly to issue new stocks in such a situation – typically due to the classical debt overhang problem [Hanson et al., 2011] and/or the adverse selection problem [Greenlaw et al., 2012] – the bank is likely to try to restore its capital ratio above 8% by selling assets, that is, by engaging in fire sales. Such undesirable consequences of capital

[1] See [BCBS, 2011], the official document on Basel III written by the Basel Committee on Banking Supervison (BCBS).

requirements that may induce banks to engage in fire sales have been recognized by academics and regulators. For example, [French et al., 2010, p46] note in their well-known collective report on the financial system that

> "because of the mark to market accounting, fire sales by some firms may force others to liquidate positions to satisfy capital requirements. These successive sales can magnify the original temporary price drop and force more sales."

while the Basel committee acknowledges ([BCBS, 2014]) that during the subprime crisis, the banking sector was forced to

> "reduce its leverage in a manner that amplified downward pressures on asset prices. This deleveraging process exacerbated the feedback loop between losses, falling bank capital and shrinking credit availability."

While fire sales and price-mediated contagion appear to play a crucial role in spreading and amplifying market shocks, the academic literature on the topic, which is less abundant compared to that on direct contagion, either analyze past fire sales episodes instead of anticipating new fire sales ([Anton and Polk, 2014, Jotikasthira et al., 2012, Khandani and Lo, 2011, Cont and Wagalath, 2016]) or consider a simple (i.e., unweighted) capital ratio instead of a risk-based capital ratio for banks ([Caccioli et al., 2014, Greenwood et al., 2015]). We thus believe that there is still a need for a risk-based capital ratio model of indirect contagion that could be *easily calibrated* to public data (i.e., contained in the annual reports of banks) to anticipate fire sales in the banking system and its consequences after a common shock. Such a framework should be of interest for regulators as a possible toolkit to draw quantitative regulatory measures such as the systemic risk capital surcharge for large banks. The seminal paper by [Greenwood et al., 2015] (see also [Duarte and Eisenbach, 2018] for a related paper) started to bridge this gap by proposing such a framework that can be calibrated to public data[2]. However, they consider a simple capital ratio, a proxy for the leverage ratio and do not address the equilibrium (fire sales) problem.

In this paper, we explicitly consider an equilibrium model of strategic interaction through price-mediated contagion only in that each bank is assumed to hold the same risky marketable security such as an exchange-traded fund. The failure of one bank thus has no direct impact on the rest of the banking system as there is no contractual link (e.g., OTC derivatives, repo...) between banks. Contagion of failures may occur but only through a price effect caused by fire sales. In the literature on the subject, as already said, it is common to assume that banks are subject to simple capital ratio instead of a RBC but also that they implement simple rules of thumbs when liquidating their assets (e.g., [Caccioli et al., 2014], [Cont and Schaanning, 2016], [Greenwood et al., 2015], see also [Feinstein, 2017]). In [Greenwood et al., 2015], it is assumed that each bank has a leverage target

[2] Technically, they use public data from the European Banking Authority, which supplement the one contained in annual reports of banks.

so that the bank trades assets when the leverage differs from the specified target. The authors note (in appendix B) that when the price impact is large enough, banks will be wiped out in a few periods but the equilibrium problem (i.e., the stationary state) is not considered. In the same vein, in [Capponi and Larsson, 2015], each bank tracks a fixed leverage target and buys or sells assets according to this objective.

We depart from these papers in that we assume that banks are subject to a risk-based capital requirement, as implemented in practice by regulators and that they liquidate *optimally* their assets, as in [Cifuentes et al., 2005] or in [Braouezec and Wagalath, 2018]. More importantly, we recognize the *strategic aspect* of the liquidation problem. When some banks liquidate a non negligible portion of their assets, this generates a "negative externality" to the other market participants through the price impact and we consider, as usual in economics, the equilibrium situation in which no bank wants to unilaterally deviate from its equilibrium selling strategy, i.e., the Nash equilibrium. Our model of fire sales actually gives rise to a *game with strategic complementarities*, initiated in economic theory by [Milgrom and Roberts, 1990] and [Vives, 1990]). It turns out to be closely related to the one of [Cifuentes et al., 2005] (see also [Chen et al., 2016]) in which they also consider an equilibrium situation but in a non-strategic framework. However, as they consider a network model à la [Eisenberg and Noe, 2001], its calibration remains difficult because information on bilateral exposures between banks are scarce [Upper, 2011], not to say unobservable. Within our framework, we make the simplifying assumption that there are no bilateral exposures between banks, that is, the unique source of systemic risk is price mediated contagion, but this allows us to easily calibrate our model to publicly-available data. Using the panel of banks considered by the American regulator to implement the 2015 regulatory stress tests (CCAR), once the parameters of each bank have been calibrated[3], we compute the Nash equilibrium associated to the game of liquidation under various scenarios (shock/price impact) and quantify the effects of price-mediated contagion in terms of insolvency, i.e., the fraction of banks that are insolvent at equilibrium. In general, the relationship between the price impact (or the shock) and the fraction of insolvent banks is *non-linear* and we quantify this non-linearity for the panel of banks under consideration. For instance, when the common shock is equal to 5%, at equilibrium, the fraction of insolvent banks is equal to 10% with a price impact of 3% while this fraction skyrockets to 30% with a price of 5%.

We then relax the one risky asset assumption and offer a two risky assets model with one marketable asset and one non-marketable asset (loans). After a shock, we make the realistic assumption that a given bank can only liquidate the marketable asset. We develop an empirical study using this more sophisticated model and focus on the six banks participating to the 2015 CCAR and with the most significant trading operations. We motivate our choice for a benchmark market shock

[3]It is interesting to note that contrary to the applied literature on game theory in IO, see for instance [Bajari et al., 2013], we make here no use of econometric methods. We use data contained in the annual reports of banks, together with our model, to imply the relevant parameters. It is similar in the spirit to the way an implied volatility is computed using the Black Scholes model together with observed price of the vanilla option.

using the result of the 2015 CCAR stress test and show that, depending on the price impact, the contagion of failures, which is not taken into account in the regulatory stress test, can be significant.

The remainder of the paper is organized as follows. Section 2 presents the one (risky) asset model. Section 3 studies the theoretical properties of the equilibrium while section 4 illustrates our results on the US banking market. Section 5 presents the two (risky) assets model and the empirical results. Section 6 concludes.

2 A benchmark framework for price-mediated contagion

2.1 Banks' balance-sheets and regulatory constraints

Consider a set $B = \{1, 2, ..., p\}$ of $p \geq 2$ banks that can invest in a risky asset and in cash. For each bank i, we denote by $v_i > 0$ the amount of cash (in dollars) and by $q_i P_t > 0$ the value (in dollars) of risky assets, where q_i is the quantity (in shares) of risky assets held by the bank and P_t is the market price of the risky asset at a given date t. Let D_i be the sum of the value of deposits and/or debt securities, typically coupon bonds, that have been issued by bank i. From limited liability of stockholders, the value of equity (or capital) at time t thus is given by:

$$E_{i,t} = \max\{A_{i,t} - D_i; 0\} = \max\{v_i + q_i P_t - D_i; 0\} \tag{2}$$

where $A_{i,t} = v_i + P_t q_i$ defines the total value of the assets of the bank. The balance-sheet of the bank at time t is as follows.

Balance-sheet of bank i at time t

Assets	Liabilities
Cash: v_i	Debt: D_i
Risky assets: $q_i P_t$	Equity: $E_{i,t}$
$A_{i,t}$	$E_{i,t} + D_i$

In practice, banks may invest in various risky securities (subject to market risk, credit risk...) so that the above balance-sheet, composed with a single risky asset, is a simplified one. We shall explain later on how to make the connexion between real balance sheets found in the registration reports of banks and our model.

Assumption 1 *The risky asset is a financial security issued by a non-financial institution whose price is quoted on financial markets.*

This marketable security is typically a stock index or an ETF replicating a stock index. It can also be a stock issued by a non-financial corporation or even a bond issued by a government or a non-financial corporation. In particular, this financial security is *not* a claim issued by a bank so that the default of a given bank has no direct impact on the rest of the banking system because there is no direct contractual links (e.g., repo, OTC derivatives...) between banks. As such, contrary to network models initiated by [Eisenberg and Noe, 2001] and [Cifuentes et al., 2005], (see [Glasserman and Young, 2016] for a recent and comprehensive overview), direct contagion of failures is not possible in our framework. The unique source of contagion is indirect, through prices. This single risky asset setting is relevant from a regulatory stress-testing perspective as it enables regulators to study worst case scenarios where banks' trading books are highly correlated. It is also motivated by the fact that banks tend to adopt similar behaviors and invest in the same risky assets or have positions in risky assets that can be considered as collinear to a common benchmark.

As discussed in section 1, banking regulation imposes banks (i.e., insured depository institutions) to hold enough capital as a *percentage* of their risk-weighted assets (RWA). Since cash is a riskless asset, it does not require any capital so that its regulatory weight is equal to zero. However, a risky asset requires some capital as a function of its risk, measured in some sense, and thus has a positive risk weight. Within our model, since there is a single risky asset, the risk-weighted asset of bank i is simply equal to

$$\mathrm{RWA}_{i,t} = \alpha_i q_i P_t \tag{3}$$

where α_i is the risk weight of bank i associated to the risky asset. Note that α_i may vary across banks because some of them make use of internal models to compute the RWAs.

Let $\theta_{i,t}$ be the risk-based capital ratio (RBC) as defined in equation (1) for a given bank i at time t:

$$\theta_{i,t} := \frac{E_{i,t}}{\mathrm{RWA}_{i,t}} \tag{4}$$

For the sake of interest, we assume that all banks are solvent at date t, that is, $E_{i,t} > 0$ for all $1 \leq i \leq p$. This means that $E_{i,t} = A_{i,t} - D_i > 0$ and the RBC of bank i can be written:

$$\theta_{i,t} = \frac{E_{i,t}}{\mathrm{RWA}_{i,t}} = \frac{A_{i,t} - D_i}{\alpha_i q_i P_t} > 0 \tag{5}$$

Denote by θ_{min} the minimum capital ratio imposed by the regulator. Since $E_{i,t}$ is the total capital, equal to Tier 1 plus Tier 2, θ_{min} thus is equal to 8%. For the sake of interest, we shall assume that, at date t, all banks comply with the regulatory constraint:

$$\theta_{i,t} \geq \theta_{min} \text{ for each } i = 1, 2, ..., p \tag{6}$$

When one inspects the balance-sheet of a universal bank (i.e., retail/investment banking), as already said, it has positions in many risky assets and not only in a single one. Since the total capital and the total risk weighted assets RWA are publicly disclosed in the annual report, it is possible to imply from the consolidated balance-sheet of each bank the weight α_i that we shall call an *implied aggregate risk weight*. From the observed balance-sheet, $A_{i,t} - v_i$ is the total value of the assets minus cash, so that it suffices to set $q_i P_t = (A_{i,t} - v_i)$ to obtain the total value of the risky asset. Since both $E_{i,t}$ and $\theta_{i,t}$ are also disclosed in the annual report of the bank, from equation (5), it thus follows that

$$\alpha_i = \frac{E_{i,t}}{\theta_{i,t}(A_{i,t} - v_i)} = \frac{\mathrm{RWA}_{i,t}}{A_{i,t} - v_i} \tag{7}$$

For most large international banks, cash is small compared to the total size of assets (typically less than 5%), which means that $A_{i,t} - v_i \approx A_{i,t}$ so that

$$\alpha_i \approx \frac{E_{i,t}}{\theta_{i,t} A_{i,t}} = \frac{\mathrm{RWA}_{i,t}}{A_{i,t}} \tag{8}$$

2.2 Impact of an exogenous shock on banks' capital ratios

Assume that a shock on the risky asset occurs at date t^+ and denote $\Delta \in (0, 1)$ the size of the adverse shock in percentage of P_t. The price of the risky asset at time t^+ thus is equal to

$$P_{t^+} = P_t(1 - \Delta) \tag{9}$$

Since each bank holds the risky asset, Δ is a common adverse shock price that could even be interpreted as a systemic shock. At time t^+, right after the shock, the balance-sheet of bank i is given as follows.

Balance-sheet at time t^+

Assets	Liabilities
Cash: v_i Risky assets: $q_i P_t(1 - \Delta)$	Debt: D_i Equity: E_{i,t^+}
A_{i,t^+}	$E_{i,t^+} + D_i$

It is the role of equity to absorb the shock, i.e., the loss which is equal to $q_i P_t \Delta$ in dollars. The RBC of bank i at date t^+ is equal to

$$\theta_{i,t^+}(\Delta) = \frac{\max\{A_{i,t^+} - D_i; 0\}}{\mathrm{RWA}_{i,t^+}} = \frac{\max\{E_{i,t} - q_i P_t \Delta; 0\}}{\alpha_i q_i P_t (1 - \Delta)} \tag{10}$$

In the remainder of this paper, we work under the following assumption.

Assumption 2 *At date t, each bank's equity is lower than the size of its risky assets, that is, for all $1 \leq i \leq p$:*

$$E_{i,t} < q_i P_t \tag{11}$$

This assumption is natural in the banking system as, in practice, banks' equities typically do not exceed 20% of their risky assets.

Lemma 1 *Under Assumption 2, the bank's RBC after the shock is a decreasing function of the shock size Δ.*

The proof is given in the Appendix. A given bank i may thus be in one of the three following situations, depending on the size of the shock Δ:

1. solvent and complying with regulatory capital requirement, that is $\theta_{i,t^+}(\Delta) \geq \theta_{min}$
2. solvent but not complying with regulatory capital requirement, that is $0 < \theta_{i,t^+}(\Delta) < \theta_{min}$
3. insolvent, that is $\theta_{i,t^+}(\Delta) = 0$, which is equivalent to $E_{i,t} - q_i P_t \Delta \leq 0$

Let us define two important thresholds:

$$\Delta_i^{sale} := \quad \inf\{\Delta \in [0,1] : \theta_{i,t^+}(\Delta) = \theta_{min}\} \tag{12}$$
$$\Delta_i^{fail} := \quad \inf\{\Delta \in [0,1] : E_{i,t^+}(\Delta) = 0\} \tag{13}$$

The following lemma characterizes those two thresholds for each bank and follows directly from equations (12) and (13):

Lemma 2 *For each bank i, the critical thresholds Δ_i^{sale} and Δ_i^{fail} defined in equations (12) and (13) can be written explicitly as follows:*

$$\Delta_i^{sale} = \frac{E_{i,t} - \alpha_i \theta_{min} q_i P_t}{q_i P_t (1 - \alpha_i \theta_{min})} = \frac{\Delta_i^{fail} - \alpha_i \theta_{min}}{1 - \alpha_i \theta_{min}} > 0 \tag{14}$$
$$\Delta_i^{fail} = \frac{E_{i,t}}{q_i P_t} > 0 \tag{15}$$
$$\text{with} \quad \Delta_i^{sale} < \Delta_i^{fail}$$

The knowledge of Δ_i^{sale} and Δ_i^{fail} enables to predict the situation of bank i after a shock and anticipate potential reactions. If $\Delta \geq \Delta_i^{fail}$, the shock will leave bank i insolvent, while if $\Delta \leq \Delta_i^{sale}$ bank i's equity will not only absorb the shock but also keep the bank's capital ratio above the minimum regulatory threshold. The interesting scenario, that we explore within this paper, occurs when $\Delta_i^{sale} < \Delta < \Delta_i^{fail}$ for some bank i: in this case, the bank is able to absorb the exogenous shock Δ but is left with a regulatory capital ratio that is lower than θ_{min}.

Since each bank is characterized by the two thresholds Δ_i^{sale} and Δ_i^{fail}, the banking system is characterized by $2p$ thresholds, i.e., by $(\Delta_i^{sale}, \Delta_i^{fail})$, $i = 1, 2, ..., p$. Without loss of generality, we shall assume that:

$$\Delta_1^{fail} \leq \Delta_2^{fail} \leq ... \leq \Delta_p^{fail} \tag{16}$$

We define now the four following thresholds:

$$\underline{\Delta}^{sale} = \inf_{i \in B} \Delta_i^{sale} \qquad \overline{\Delta}^{sale} = \sup_{i \in B} \Delta_i^{sale} \tag{17}$$
$$\underline{\Delta}^{fail} = \inf_{i \in B} \Delta_i^{fail} = \Delta_1^{fail} \qquad \overline{\Delta}^{fail} = \sup_{i \in B} \Delta_i^{fail} = \Delta_p^{fail} \tag{18}$$

2.3 Endogenous fire sales and feedback effects

Since Δ is a common shock, it affects the balance-sheet of *all* banks that hold the risky asset and may leave some of them undercapitalized. Banks that do not comply with the regulatory capital constraints may restore their capital ratio above the minimum required θ_{min} in two main ways.

1. They may issue new shares and hence increase the numerator of the risk-based capital ratio.

2. They may also sell assets and decrease the denominator of the risk-based capital ratio.

After such a common shock, e.g., what happened during the subprime crisis, it may be difficult for such banks to sell new stocks. In such a situation, the preferred solution for a given bank i to restore its regulatory capital ratio back above the minimum required is to *sell assets*. In line with the existing literature on the subject (e.g., [Brunnermeier and Oehmke, 2014, Cifuentes et al., 2005, Greenlaw et al., 2012, Greenwood et al., 2015]) and consistent with observed behavior of banks, we make the assumption that undercapitalized banks can only engage in asset sales in order to restore their capital ratio. Within our study, we define such forced sales as "fire sales". As in most models, e.g., [Elliott et al., 2014, Caccioli et al., 2014], when a bank is unable to restore its capital ratio above θ_{min}, we assume that it is fully liquidated at date $t+1$.

In practice, banks are sometimes able to issue stocks despite the situation of distress. To give a recent example, following its problems with the American justice, Deutsche Bank decided in March 2017 to issue 687.5 million stocks for a total value of \$8 Billion, which is approximately equal to the amount of the fine imposed by the US Justice (\$7.2 Billion). It is important to point out at this stage that it wouldn't be difficult to introduce in our model the possibility for each bank to recapitalize up to a certain amount, say (at most) a given fraction of their existing total capital. In such a case, after a large shock, if the recapitalization is not sufficient for the bank to comply with the regulatory capital ratio, the unique possibility for the banks is to sell assets and we are back to our model. Allowing each bank to recapitalize up to a certain proportion of its existing total capital would thus only change the two thresholds Δ_i^{sale} and Δ_i^{fail}.

We denote by $x_i \in [0,1]$ the proportion of risky assets sold by bank i at date $t+1$, in reaction to the shock Δ at date t^+. When bank i does not need to liquidate assets, then $x_i = 0$. On the contrary, when the shock Δ is such that bank i is insolvent or unable to restore its capital ratio above θ_{min}, then it is fully liquidated and $x_i = 1$. The volume (in shares) of liquidation by bank i is denoted by $x_i q_i$ and $\sum_{i \in B} x_i q_i$ denotes the total volume of fire sales in the banking system at date $t+1$.

Fire sales obviously impact the price of the asset at date $t+1$ and we assume here this price impact to be linear. We introduce the asset market depth Φ which is a linear measure of the asset liquidity [Kyle and Obizhaeva, 2016]. In [Cont and Wagalath, 2016], it is shown that the relevant quantity to capture the magnitude of feedback effects is $\frac{\sum_{i \in B} q_i}{\Phi}$. The greater this parameter, the greater the size of the banking system compared to asset market depth and the greater the feedback on the asset price in the presence of fire sales. The asset price at date $t+1$ thus depends on the vector of liquidations $\boldsymbol{x}(\Delta, \Phi) := \boldsymbol{x} = (x_1, x_2, ..., x_p) \in [0,1]^p$, and this vector of liquidation depends on both the shock Δ and the market depth Φ.

Assumption 3 *The price of the risky asset at time $t+1$ is equal to*

$$P_{t+1}(\boldsymbol{x}, \Phi) = P_t\,(1-\Delta)\left(1 - \frac{\sum_{i \in B} x_i q_i}{\Phi}\right) \tag{19}$$

$$\frac{Q_{tot}}{\Phi} < 1 \tag{20}$$

$$\text{where } Q_{tot} = \sum_{i \in B} q_i \tag{21}$$

In a more general model, one could allow the price impact to be a non linear function of the quantity sold. For instance, it could be possible, as in [Cifuentes et al., 2005], to consider a convex price impact function in which $P_{t+1}(\boldsymbol{x}, \Phi) = P_t\,(1-\Delta)\, e^{-\frac{\sum_{i \in B} x_i q_i}{\Phi}}$. This would only complicate the analysis without adding new economic insight. In any event, as we shall see later on, the proof of the existence of the equilibrium would also work in the non-linear case. The assumption that $\frac{Q_{tot}}{\Phi} < 1$ is aimed at keeping the price strictly positive even if all banks fully liquidate their positions on the risky asset.

As discussed above, the market depth Φ is a linear measure of the risky asset's liquidity: the larger this parameter, the more liquid the asset. When $\Phi = \infty$, the asset is infinitely liquid and fire sales do not impact the asset price. In practice, roughly speaking, the asset liquidity is measured by the presence of buyers *outside* the banking system when banks need to sell the risky assets. Such potential buyers may typically be hedge funds that can absorb fire sales from banks and limit their impact on asset prices. As such, our model shows the (relative) importance of the shadow banking as a possible stabilizing force in the case of a banking crisis. Note however that hedge funds might also face funding difficulties during a banking crisis, which may limit their ability to buy back assets from distressed banks [Caballero and Simsek, 2013]. In our model, this kind of difficulty would be translated into a lower Φ. In a dynamic model, Φ could be time-dependent and an evaporation of the liquidity would be modeled by a sharp fall of Φ.

At time $t+1$, the balance-sheet of bank i that sold a portion x_i of the risky asset is given below:

Balance-sheet of bank i at date $t+1$ after deleveraging

Assets	Liabilities
Cash: $v_i + x_i q_i P_{t+1}(\boldsymbol{x}, \Phi)$	Debt: D_i
Risky asset: $(1-x_i) q_i P_{t+1}(\boldsymbol{x}, \Phi)$	Equity: $E_{i,t+1}$
$A_{i,t+1} = v_i + q_i P_{t+1}(\boldsymbol{x}, \Phi)$	$E_{i,t+1} + D_i$

where $P_{t+1}(\boldsymbol{x}, \Phi)$ is given in Assumption (3). Let $E_{i,t+1}(\boldsymbol{x})$ be the total capital at time $t+1$ after deleveraging. From the above balance-sheet, it is not difficult to show that

$$E_{i,t+1}(\boldsymbol{x}, \Delta) = \max\left\{ E_{i,t} - q_i P_t \left(\Delta + \frac{\sum_{j \in B} x_j q_j}{\Phi}(1-\Delta) \right); 0 \right\} \tag{22}$$

and note that it is a *decreasing* function of x_i due to the existence of a price impact. The regulatory capital ratio of bank i at time $t+1$ (i.e., after deleveraging) thus is equal to

$$\theta_{i,t+1}(\boldsymbol{x}, \Delta) = \frac{E_{i,t+1}(\boldsymbol{x})}{\alpha_i q_i P_{t+1}(\boldsymbol{x}, \Phi)(1-x_i)} \tag{23}$$

with the natural convention that $\theta_{i,t+1}(\boldsymbol{x}, \Delta) = 0$ when $x_i = 1$ and when $E_{i,t+1} = 0$. Let us now introduce the concept of the *implied shock*:

$$\Delta(\boldsymbol{x}) := \Delta + \frac{\sum_{j \in B} x_j q_j}{\Phi}(1-\Delta) \tag{24}$$

associated to the vector of liquidation $\boldsymbol{x}$ such that the price of the risky asset at date $t+1$ can be written as follows

$$P_{t+1}(\boldsymbol{x}, \Phi) = P_t(1 - \Delta(\boldsymbol{x})) \tag{25}$$

As long as $\boldsymbol{x} \neq \mathbf{0}$ (i.e., there is at least one bank engaging in fire sales) and $\Phi < \infty$ (i.e., the asset is not infinitely liquid), $\Delta(\boldsymbol{x}) > \Delta$ so that fire sales at date $t+1$ actually reinforce the underperformance of the asset caused by the initial shock Δ at date t^+. By re-inserting the implied shock in equation (23) and by dividing the numerator and the denominator by $q_i P_t$, equation (23) reduces to

$$\theta_{i,t+1}(\boldsymbol{x}, \Delta) = \frac{\max\{\Delta_i^{fail} - \Delta(\boldsymbol{x}); 0\}}{\alpha_i(1-x_i)(1-\Delta(\boldsymbol{x}))} \tag{26}$$

so that we immediately obtain the following equivalence.

$$E_{i,t+1}(\boldsymbol{x}, \Delta) > 0 \Longleftrightarrow \Delta_i^{fail} - \Delta(\boldsymbol{x}) > 0 \tag{27}$$

Assumption 4 *Each bank $i = 1, 2, ..., p$ rebalances its portfolio of assets (i.e., deleverage) in order to minimize $x_i \in [0,1]$ subject to the constraint*

$$\theta_{i,t+1}(\boldsymbol{x}, \Delta) \geq \theta_{min} \tag{28}$$

If the constraint can not be satisfied for some $x_i \in [0, 1)$, then bank i is insolvent and is costlessly liquidated at time $t+1$ so that $x_i = 1$.

It is usual to call our framework a game in strategic (or normal) form, which is characterized by three elements (see e.g., [Fudenberg and Tirole, 1991] chapter 1); the set of players, here the set of p banks, the pure-strategy spaces $X_i := [0,1]$ for each bank $i = 1, 2, ..., p$, and payoff functions $u_i(\boldsymbol{x})$ for each profile $\boldsymbol{x} = (x_1, ..., x_2)$. Within our framework, since we consider a static game, all decisions are taken at time $t+1$, that is simultaneously.

Let $\boldsymbol{x} = (x_i, x_{-i})$ where $x_{-i} \in [0,1]^{p-1}$ is a $p-1$-dimensional vector. To understand the motivation for assumption 4, assume that x_{-i} is *given* and *known* from bank i. From the balance sheet of a given bank i after deleveraging at time $t+1$, the total value of the assets of bank i is equal to $A_{i,t+1}(x_i, x_{-i}) = v_i + q_i P_{t+1}(x_i, x_{-i}, \Phi)$ and is a decreasing function x_i. For a given amount of cash v_i, it is fair to think that the expected profit of each bank i, denoted $\mathbb{E}\pi_i$, is a fraction of the total value of the assets, that is, $\mathbb{E}\pi_i(x_i, x_{-i}) = \gamma_i A_{i,t+1}(x_i, x_{-i}) := u_i(x_i, x_{-i})$ for some $\gamma_i < 1$ which may depend upon v_i. The expected profit of bank i thus is also a decreasing function of x_i. Assumption 4 just says that, given x_{-i}, each bank tries to maximize its expected profit by choosing the *smallest* $x_i \in [0,1]$ such that $\theta_{i,t+1}(x_i, x_{-i}, \Delta) \geq \theta_{min}$. Of course, x_{-i} is unknown from bank i. As usual in game theory, we consider in this paper a Nash equilibrium (in pure strategy), which is defined as the vector of strategy $\boldsymbol{x}^* = (x_i^*, x_{-i}^*)$ such that no bank wants to *unilaterally deviate* from this strategy. Put it differently, from the observation of the vector $\boldsymbol{x}^* = (x_i^*, x_{-i}^*)$, the best thing to for each bank i, $i = 1, 2, ..., p$ is precisely to sell a fraction equal to x_i^*.

Throughout the paper, we make the assumption that the market depth Φ is "large enough". By this, we mean that, for the highest shock size Δ considered, it is in the interest of solvent banks, i.e., banks that are not wiped out, to sell a positive fraction of their asset in order to (try to) restore their risk-based capital ratio back above the minimum required.

Let $S_\Delta = \{i \in B : \Delta < \Delta_i^{fail}\}$. When there is no price impact, i.e., when $\Phi = \infty$, each bank $i \in S_\Delta$ will be able to sell smallest fraction $\underline{x}_i < 1$ such that $\theta_{i,t+1}(\underline{x}_i, x_{-i}, \Delta) = \theta_{min}$ since $\theta_{i,t+1}(x_i, x_{-i}, \Delta)$ is an increasing function of x_i for any x_{-i}. However, when there is a positive price impact, i.e., $\Phi < \infty$, $\theta_{i,t+1}(x_i, x_{-i}, \Delta)$ may not be an increasing function for all x_{-i}. For the sake of interest, we make the following *sufficient condition*

$$\frac{\partial \theta_{i,t+1}}{\partial x_i}(0, \mathbf{1}, \Delta) > 0 \quad \forall i \in S_\Delta \tag{29}$$

where $\mathbf{1} = (1, ..., 1)$ is a $p-1$ dimensional vector[4]. In words, each bank i in S_Δ is such that if all the other banks would liquidate 100% of their assets, the risk-based capital ratio of bank i would still be an increasing function of x_i in zero so that it is always in the interest of bank i to try to liquidate a positive quantity. Of course, this condition does not guarantee the existence of a solution for bank i. Since the risk-based capital ratio of each bank i is a continuous function of Φ, there exists a smallest Φ for which the sufficient condition is satisfied.

[Caccioli et al., 2014], [Cont and Schaanning, 2016] and [Greenwood et al., 2015] also consider a liquidation problem with price impact but in which each bank is assumed to make use of a purely mechanical decision rule to liquidate its assets (e.g., the proportional liquidation rule). As a result, as observed in [Caccioli et al., 2014], the sequence of liquidation over time becomes closely related to contagion processes in epidemiology. In this paper, we follow a different approach since we explicitly

[4]This condition is clearly sufficient because it needs only to hold at equilibrium, and not when all the banks except bank i sell 100% of their assets.

recognize the strategic feature of the liquidation problem. From equation (23), the RBC of a given bank i is influenced by the decisions of *all* banks so that a given bank can not decide *independently* of the other banks the minimum portion of the risky asset to liquidate. The problem is similar to a Cournot oligopoly but somehow more complex as each bank is explicitly faced with a regulatory constraint. Since all liquidation decisions are taken at time $t+1$ only, we look for a static Nash equilibrium.

When a corporation such as a bank (or even more generally a non financial institution) is liquidated, one can identify two main types of costs associated with the failure of that institution [Glasserman and Young, 2016]. Administrative and legal costs called bankruptcy costs on the one hand and costs of delay in making payments on the other hand. In (banking) network models, everything else equal, the greater the bankruptcy costs when bank i fails, the lower the recovery rate of its claimants (e.g., the other banks in the banking system). As a result, bankruptcy costs increase the magnitude and the likelihood of failure cascades. Within our model, as there is no direct links between banks, bankruptcy costs *do not affect* the likelihood nor the magnitude of the contagion process. Since our aim is to focus on failure contagion, without loss of generality, we assume that these bankruptcy costs are equal to zero. Of course, if our aim was to analyze the cost borne say by depositors (or bondholders) in case of failure of banks, then, bankruptcy costs would play an important role. Note interestingly that in Europe, it is actually the role of the recent single resolution mechanism, the second pillar of the banking union “to ensure the efficient resolution of failing banks with minimal costs for taxpayers and to the real economy[5]”. It is also stated that the resolution of a bank could be done over a week-end...

Within our framework, since each bank is assumed to be exposed to a security issued by a non-financial institution, as already discussed, there is no network of interconnections between banks. As a result, a given bank only needs to know the balance sheet of the other banks and the price impact to determine whether or not it may be indirectly impacted by fire sales. Given the various information that are publicly disclosed in the annual reports of banks, we assume that information is complete.

Assumption 5 *Information is complete.*

Let $I_i = (D_i, q_i, A_i, E_i, v_i, \mathrm{RWA}_i, RBC_i)$ be the information disclosed in the annual report of a given bank i. When a bank j knows I_i from the annual report of bank i, since the current market price is observable, bank j is able to determine α_i and the two thresholds Δ_i^{sale} and Δ_i^{fail}. If one defines the structure of the game as $\{(I_i)_{i=1}^{p}; \Phi\}$ together with the decision rules, complete information is usually defined by saying that the structure of the game is common knowledge.

[5]See https://www.bankingsupervision.europa.eu/about/bankingunion/html/index.en.html

3 Strategic fire sales: Nash equilibrium with strategic complementarities

The timing of the *liquidation game* is by assumption as follows.

1. Right after the shock, the price of the risky asset decreases by a percentage $\Delta > 0$.
2. At time $t+1$, each bank $i = 1, ..., p$ liquidates a portion $x_i \in [0,1]$ of the risky asset, so that the implied shock is equal to $\Delta(\boldsymbol{x}) = \Delta + (1-\Delta)\left(\frac{\sum_{i\in B} x_i q_i}{\Phi}\right)$.

3.1 Best responses and upward jumps

Recall that $\boldsymbol{x} = (x_i, x_{-i})$ where $x_{-i} \in [0,1]^{p-1}$ is a $p-1$-dimensional vector and let us write the vector of liquidation as $\boldsymbol{x} = (BR_i(x_{-i}), x_{-i})$, where $BR_i(x_{-i})$ is the *unique best response* of bank i given $x_{-i} \in [0,1]^{p-1}$ in the sense of the minimization problem given in assumption 4. Within our model, as there is no direct link between banks, each bank is impacted by the decision of all the other banks only through the price impact. Let

$$\mathcal{S}_i^v = \{x_{-i} \in [0,1]^{p-1} : \sum_{i\neq j} q_j x_j = v\} \tag{30}$$

From equation (24), since the implied shock depends on $x_{-i} \in [0,1]^{p-1}$ only through the sum of its components $\sum_{i\neq j} q_j x_j$, for bank i, each $x_{-i} \in \mathcal{S}_i^v$ yields the *same* unique best response. As a result, for any shock $\Delta > 0$, as long as $x_{-i} \in \mathcal{S}_i^v$, the unique best response of bank i can be written as a function of v, i.e., $x_i^* = BR_i(v, \Delta)$. Assume that $x_{-i} \in \mathcal{S}_i^v$ for some $v > 0$. The implied shock defined in (24) thus can be written as a function of two variables x_i and v.

$$\Delta(x_i, v) = \Delta + (1-\Delta)\left(\frac{x_i q_i + v}{\Phi}\right) \tag{31}$$

and is an *increasing function* of v and of x_i. Let $BR_i(v, \Delta)$ be the best response of bank i and note that this best response also depends on Φ.

Lemma 3 *1. For a given Δ and for each $i = 1, ..., p$, if $v_2 \geq v_1$, then, $BR_i(v_2, \Delta) \geq BR_i(v_1, \Delta)$.*

2. Let $v > 0$. For each $i = 1, ..., p$, if $\Delta_2 \geq \Delta_1$, then, $BR_i(v, \Delta_2) \geq BR_i(v, \Delta_1)$.

Proof. See the appendix.

Part 2 of the above lemma says, as one can expect, that each bank needs to sell more risky assets when the shock Δ is larger, everything else equal. In the same vein, part 1 states that the best response of a given bank i increases with v, which means that bank i has an incentive to liquidate more risky assets when the other banks increase their volume of fire sales. In economic theory, this property of monotone increasing best response is called *strategic complementarity* [Vives, 1990]. We shall now show that the best response needs not be a continuous function of v.

Lemma 4 *If there exists $\tilde{v}$ such that*

$$\sup_{x_i \in [0,1]} \theta_{i,t+1}(x_i, \tilde{v}, \Delta) \geq \theta_{min} \tag{32}$$

and for all $\epsilon > 0$:

$$\sup_{x_i \in [0,1]} \theta_{i,t+1}(x_i, \tilde{v} + \epsilon, \Delta) < \theta_{min} \tag{33}$$

then $BR_i(v, \Delta)$ is not a continuous function of v and exhibits an upward jump for $v = \tilde{v}$.

The proof is straightforward: if $\sup_{x_i \in [0,1]} \theta_{i,t+1}(x_i, \tilde{v}, \Delta) \geq \theta_{min}$, then this means that $BR_i(\tilde{v}, \Delta) < 1$. In addition, if for all $\epsilon > 0$, $\sup_{x_i \in [0,1]} \theta_{i,t+1}(x_i, \tilde{v} + \epsilon, \Delta) > \theta_{min}$, this means that bank i has to fail and so $BR_i(\tilde{v} + \epsilon, \Delta) = 1$ for all $\epsilon > 0$. As such, the function $BR_i(v, \Delta)$ exhibits an upward jump for $v = \tilde{v}$ and is not continuous in v. This is illustrated in figure 1.

In the classical proof of the existence of a Nash equilibrium, the application of a fixed point theorem such as the Brower's one requires the best response to be continuous. However, as noticed in their early paper, [Roberts and Sonnenschein, 1976] proved the existence of the Nash equilibrium assuming that the discontinuities take the form of upward jumps (see [Vives, 1990] section 7 or [Milgrom and Roberts, 1994] p. 447 for a discussion). Within our framework, depending on the parameters, as we have seen, one can not exclude such upward jumps, which means that a more powerful result than the classical Brouwer's fixed point theorem should be used.

3.2 Existence of a Nash equilibrium

We first start by giving the definition of a Nash equilibrium in our model.

Definition 1 *For a given initial shock $\Delta > 0$, the vector of liquidation $\boldsymbol{x}^* = (x_1^*, ..., x_p^*) \in [0,1]^p$ is a Nash equilibrium if and only if for all $i = 1, 2, ..., p$:*

$$BR_i(x_{-i}^*, \Delta) := x_i^* = \min\left\{x_i \in [0,1) \;\; \textit{such that}\; \theta_{i,t+1}(x_i, x_{-i}^*, \Delta) \geq \theta_{min}\right\} \;\; \textit{or} \quad x_i^* = 1 \tag{34}$$

Saying that $\boldsymbol{x}^*$ is a Nash equilibrium means that, for each $i = 1, 2, .., p$, the best response $BR_i(x_{-i}^*, \Delta)$ is equal to x_i^*. At equilibrium, the *implied shock* $\Delta(\boldsymbol{x}^*) \equiv \Delta^*$ thus is equal to

$$\Delta^* = \Delta + \left(\frac{\sum_{i \in B} x_i^* q_i}{\Phi}\right)(1 - \Delta) \tag{35}$$

To prove the existence of a Nash equilibrium within our framework, we shall use a fixed point result that requires some basic preliminaries. Recall that $X = [0,1]^p$ is the set of all liquidation vectors and consider now the pair $(X, \geq)$ where $\boldsymbol{x} \leq \boldsymbol{y} \iff x_i \leq y_i$ for each $i = 1, ..., p$ so that $(X, \geq)$ is a partially ordered set (poset for short). A poset $(X, \geq)$ is said to be a lattice if, for any pair of elements $\mathbf{x}$ and $\mathbf{y}$ of X, the supremum $\sup\{\mathbf{x}, \mathbf{y}\}$ and the infimum $\inf\{\mathbf{x}, \mathbf{y}\}$ exist in X. The lattice is said to be complete if, for all non-empty subset $E \subset X$, the supremum $\sup E$ and the infimum $\inf E$

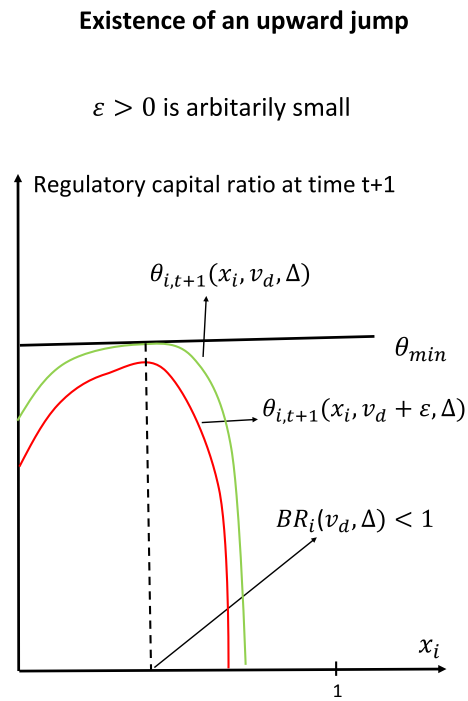

Figure 1: The best response may exhibit a discontinuity

exist in X. See for instance [Tarski, 1955],[Vives, 1990], [Milgrom and Roberts, 1990]. When X is the product of p compact sets, i.e., $X = [0,1]^p$, it is well-known, and easy to see, that the poset is a complete lattice. The following result is due to [Tarski, 1955] and, to the best of our knowledge, has been introduced in economic theory by [Vives, 1990] and [Milgrom and Roberts, 1990]. It has been subsequently used in finance by [Eisenberg and Noe, 2001] in their influential paper on systemic risk.

Tarski's theorem ([Tarski, 1955], see also [Vives, 1990] or [Milgrom and Roberts, 1990]). *Let $(L, \geq)$ be a complete lattice and f a non decreasing function from L to L and $\mathcal{F}$ the set of fixed points of f. Then, $\mathcal{F}$ is non-empty and $(\mathcal{F}, \geq)$ is a complete lattice. In particular, $sup_{\boldsymbol{x}}\, \mathcal{F}$ and $inf_{\boldsymbol{x}}\, \mathcal{F}$ belong to $\mathcal{F}$.*

As observed by [Vives, 2001] in his well-known textbook on oligopoly pricing, this fixed point result is interesting as it does not make any use of topological properties such as compactness or continuity. It only requires the function f to be non-decreasing, which is the case in our framework. The linearity of the price impact function thus plays no role in the proof of the proposition below and a non-linear price impact function would not change the existence result.

In our context, Nash equilibria will be fixed points of the function f defined by:

$$f(x) = (BR_1(x_{-1}), ..., BR_p(x_{-p})) \tag{36}$$

where $BR_i(x_{-i})$ is the unique best response of bank i given $x_{-i} \in [0,1]^{p-1}$ in the sense of the minimization problem given in assumption 4. The following proposition shows that there always exists at least one Nash equilibrium, i.e., f has at least one fixed point.

Proposition 1 *For all initial shock* $\Delta \in (0,1)$ *and market depth* $\Phi > 0$*, the set of Nash equilibria denoted* $\mathcal{F}_\Delta$ *is not empty. For any equilibrium* $\boldsymbol{x}^* \in \mathcal{F}_\Delta$*, the subset of banks that are solvent and insolvent, denoted* S^* *and* D^* *respectively, when non empty, are composed with consecutive integers, i.e., there exists* $0 \leq i(\boldsymbol{x}^*) := i^* \leq p$ *such that* $D^* = \{1, ..., i^*\}$ *and* $S^* = \{i^* + 1, ..., p\}$.

Proof. See the appendix.

Note importantly that throughout this paper, when $\mathcal{F}_\Delta$ contains more than one Nash equilibrium, we shall always consider the smallest one, that is, the one that minimizes the implied shock $\Delta(\boldsymbol{x}^*)$, or equivalently the total amount liquidated equal to the scalar product $\boldsymbol{x}^*\boldsymbol{q}$.

It is well-known that games with strategic complementarities can have more than one Nash equilibrium which may be Pareto ranked (e.g., [Vives, 2005]) and our model is of no exception. From a financial point of view, we claim that it makes only sense to consider the *smallest* Nash equilibrium. Assume for the discussion that $\mathcal{F}_\Delta = \{\boldsymbol{x}^*, \boldsymbol{y}^*\}$. Since these two equilibria are ordered, say $\boldsymbol{y}^* \geq \boldsymbol{x}^*$, it is in the interest of *all market participants* – banks, depositors, bondholders – to choose the Nash equilibrium that minimizes the market impact since it also minimizes the number of insolvent banks. As noted by [Fudenberg and Tirole, 1991] in their well-known textbook on game theory, choosing a particular equilibrium relies on some mechanism that leads all the banks to expect the same equilibrium, which thus becomes the focal point. In general, the explanation of this choice is based on *preplay communication*, that is, on the possibility for the banks to "talk" before the game. As usual in the literature, when Nash equilibria are Pareto ranked, we make here the assumption that banks are able to coordinate on the Pareto-dominant equilibrium[6], that is, on the smallest Nash equilibrium within our model. Since supervisors monitor financial stability and are reluctant to let large institutions fail, the smallest Nash equilibrium is clearly also the preferred solution of regulatory public institutions. In such a case, since each bank expects the rest of the banking sector to choose the smallest one, it is naturally a focal point.

3.3 Case of no price impact

In this section, we assume that there is no price impact, that is, $\frac{1}{\Phi} = 0$. In this case, the liquidation problem turns out to be very simple since the problem is not anymore strategic. In such a situation,

[6] In [Fudenberg and Tirole, 1991] paragraph 1.2.4 entitled *Multiple Nash equilibria, Focal points and Pareto Optimality* , they offer an example of a particular game in which players may not choose the Pareto dominant equilibrium but rather the risk dominant one, which is Pareto dominated.

when $\Delta \in (\Delta_i^{sale}, \Delta_i^{fail})$, after deleveraging, the RBC of bank i is equal to

$$\theta_{i,t+1}(x_i) = \frac{E_{i,t} - q_i \Delta P_t}{\alpha_i q_i P_t (1-\Delta)(1-x_i)} \tag{37}$$

and the numerator of equation (37) is *invariant* with respect to x_i. Since the denominator is a decreasing function of x_i that converges to zero, there exists a unique solution $\underline{x}_i^* < 1$ such that $\theta_{i,t+1}(\underline{x}_i^*) = \theta_{min}$. The solution is equal to

$$\underline{x}_i^* = 1 - \left[\frac{\Delta_i^{fail} - \Delta}{\alpha_i (1-\Delta)\theta_{min}}\right] < 1 \tag{38}$$

The following proposition is a direct consequence of the above.

Proposition 2 *In the absence of price impact, the optimal portion of the risky asset to liquidate for each bank $i = 1, ..., p$ as a function of Δ can be written as:*

- *If $\Delta \leq \Delta_i^{sale}$, then $\underline{x}_i^* = 0$*
- *If $\Delta_i^{sale} < \Delta < \Delta_i^{fail}$, then*

$$\underline{x}_i^* = 1 - \left(\frac{1 - \Delta_i^{sale}}{1-\Delta}\right)\left(\frac{\Delta_i^{fail} - \Delta}{\Delta_i^{fail} - \Delta_i^{sale}}\right) \tag{39}$$

- *If $\Delta \geq \Delta_i^{fail}$ then $\underline{x}_i^* = 1$*

Denote $x_+ := \max\{x; 0\}$ and recall that the payoff of a call option with strike price K when the underlying asset is equal to x is equal to $(x-K)_+$. For a given bank i, the amount liquidated (in dollars) is equal to $\underline{x}_i^* q_i P_t (1-\Delta)$ and, given equation (39), the amount liquidated is equal to:

$$q_i P_t \left(\frac{1 - \Delta_i^{fail}}{\Delta_i^{fail} - \Delta_i^{sale}}\right) \underbrace{\left[(\Delta - \Delta_i^{sale})_+ - (\Delta - \Delta_i^{fail})_+\right]}_{\text{Difference between two call options}} - q_i P_t (\Delta - \Delta_i^{fail})_+ \tag{40}$$

This amount liquidated can actually be expressed as the *difference between two call options*, where Δ plays the role of the underlying asset and the thresholds Δ_i^{sale} and Δ_i^{fail} the role of the strike prices. Equation (40) thus shows that the amount liquidated is the difference of two convex functions. This means that the amount liquidated is (highly) convex when Δ is close to Δ_i^{sale}. We provide empirical examples of this convexity effect in the empirical section of this paper.

3.4 Case with positive price impact: characterization of Nash equilibrium for a small shock

Let us now characterize the Nash equilibrium in the case of a *small shock*, when all banks sell a portion of the risky asset and survive the deleveraging process, that is, when for all $1 \leq i \leq p$:

$$\Delta_i^{fail} > \Delta(\boldsymbol{1}) \iff E_i - q_i P_t \Delta - \frac{Q_{tot}}{\Phi} q_i P_t (1-\Delta) > 0 \tag{41}$$

where $\boldsymbol{1} := (1, ..., 1)$. The above inequality is equivalent to:

$$\Delta < \frac{\Delta_i^{fail} - \frac{Q_{tot}}{\Phi}}{1 - \frac{Q_{tot}}{\Phi}} \tag{42}$$

and note that the rhs of equation (42) is positive for bank 1 (and thus for all banks) when Φ high enough. Recall that x_i^* is the optimal portion of risky asset liquidated by bank i (at equilibrium) when there is a positive price impact, i.e., when $\frac{1}{\Phi} > 0$. It should be clear that for each $i = 1, ..., p$, $x_i^* \geq \underline{x}_i^*$. Let $Q^* = \sum_{i=1}^{p} x_i^* q_i$ be the total amount liquidated when the price impact is positive and let $\underline{Q}^* = \sum_{i=1}^{p} \underline{x}_i^* q_i$ be this total amount liquidated when there is no price impact, where $\underline{x}_i^*$ is given by proposition 2. Since $x_i^* \geq \underline{x}_i^*$ as long as the price impact is positive, the characterization of $\delta x_i^* := x_i^* - \underline{x}_i^*$ would be a valuable result to (better) understand the sensitivity of δx_i^* as a function of the parameters of our model and hence quantify, in a tractable manner, the increase in fire sales due to feedback effects. It is precisely the aim of the following proposition to provide such a characterization. Recall that $\underline{\Delta}^{fail}$ and $\overline{\Delta}^{sale}$ are defined respectively in equations (17) and (18). In the following proposition, given the shock, it is assumed that each bank needs to sell a positive portion of the risky asset but also that each bank survived the deleveraging process (at equilibrium). As a result, at time $t+1$, $\theta_{i,t+1}(.) = \theta_{min}$ for all $1 \leq i \leq p$ and this property allows us to make further computation.

Proposition 3 *Assume that the initial shock $\Delta > 0$ is such that:*

$$\overline{\Delta}^{sale} < \Delta < \frac{\underline{\Delta}^{fail} - \frac{Q_{tot}}{\Phi}}{1 - \frac{Q_{tot}}{\Phi}} \tag{43}$$

Then

$$x_i^* = \underline{x}_i^* + \left(\frac{\underline{Q}^*}{\Phi} \frac{1 - \Delta_i^{fail}}{\theta_{min}\alpha_i(1-\Delta)} \right) + o\left(\frac{1}{\Phi}\right) \tag{44}$$

so that the total quantity sold is equal to

$$Q^* = \underline{Q}^* \times \left(1 + \frac{1}{\Phi} \sum_{i=1}^{p} \frac{q_i(1 - \Delta_i^{fail})}{\theta_{min}\alpha_i(1-\Delta)} \right) + o\left(\frac{1}{\Phi}\right) \tag{45}$$

Proof. See the appendix.

In the case of a small shock, the above proposition shows that when all the banks remain solvent after the deleveraging process, x_i^* can be expressed as the sum of $\underline{x}_i^*$, the optimal portion to liquidated when there is no price impact, and a positive quantity which depends on the parameters of the model only. From equation (44), everything else equal, when the size of the shock Δ increases, the optimal portion of the risky asset liquidated by each bank i increases while when θ_{min} (or α_i) increases, it decreases. Finally, as expected, x_i^* tends to $\underline{x}_i^*$ in the limiting case in which Φ tends to infinity.

3.5 Price mediated-contagion and amplification effect of a marginal shock

Within our model, due to price impact, the liquidation process of a given bank may adversely impact the capital ratio of other banks, which may ultimately lead to the failure of some of them through price-mediated contagion. Because of the heterogeneity in banks' capital structures and regulatory weights, captured by the distribution of thresholds $(\Delta_i^{sale}, \Delta_i^{fail})_{1\le i\le p}$, the fraction of insolvent banks is a non-linear function of the shock size even when there is no price impact. To correctly measure the amplification effect associated to a marginal shock, we thus consider the no price impact case as a benchmark. Assume for instance that the marginal shock is equal to 100 bps. If the resulting marginal fraction of insolvent banks is equal to 5% when there is no price impact, one can only say that there is an amplification effect in the positive price impact case if the marginal fraction of insolvent banks is *greater than* 5%.

Assume that the initial shock is such that $\Delta = \Delta_j^{fail}$ for some $j \in \{1, 2, ..., p-1\}$ and denote $\lambda(\Delta^*(\Delta, \frac{1}{\Phi})) := \lambda^*(\Delta, \frac{1}{\Phi})$ the fraction of insolvent banks *at equilibrium*. When there is no price impact, that is $\frac{1}{\Phi} = 0$, the subset of banks that are insolvent is equal to $\{1, ..., j\}$ so that $\lambda^*(\Delta, 0) = \frac{j}{p}$. When the initial shock increases by $\delta \ge 0$, that is, the shock is now equal to $\Delta' := \Delta + \delta$, the fraction of insolvent banks increases and is now equal to $\lambda^*(\Delta + \delta, 0) = \frac{m}{p}$, for some $m \in \{j, ..., p-1\}$. The marginal fraction of insolvent banks due to the marginal shock δ is positive and equal to $\lambda^*(\Delta + \delta, 0) - \lambda^*(\Delta, 0) = \frac{m-j}{p} \ge 0$. Of course, this quantity is positive not because of price-mediated contagion due to the existence of a positive price impact, but simply because a larger shock, depending on the distribution of the threshold, triggers additional failures. To measure the possible amplification of the marginal fraction of insolvent banks due to the existence of a positive price impact, it is thus natural to consider the following ratio

$$\mathcal{I}\left(\Delta, \delta, \frac{1}{\Phi}\right) = \frac{\lambda^*(\Delta + \delta, \frac{1}{\Phi}) - \lambda^*(\Delta, \frac{1}{\Phi})}{\lambda^*(\Delta + \delta, 0) - \lambda^*(\Delta, 0)} \ge 0 \tag{46}$$

which is indeed a ratio of two slopes. We shall say that there is an *amplification effect* at $(\Delta, \delta, \frac{1}{\Phi})$ if $\mathcal{I}\left(\Delta, \delta, \frac{1}{\Phi}\right) > 1$. Let $\delta = 100$ basis points and assume that $\mathcal{I}(.) = 2$. This means that the marginal fraction of insolvent banks when there is a price impact is *doubled* compared to the case without price impact[7]. Note that when this measure of amplification is greater than one, it may be thought of as a particular measure of *financial fragility* (of a financial system) as defined in [Allen and Gale, 2004].

Note that it may of course be the case that $\mathcal{I}(.) < 1$, and indeed equal to zero. To see this, let $\Delta_{Syst}(\frac{1}{\Phi})$ be the *smallest shock* $\Delta \le \overline{\Delta}^{fail}$ such that all the banks are insolvent at equilibrium, , i.e., $x_i^* = 1$ for each $i = 1, ..., p$. Assume that $\Delta_{Syst} < \overline{\Delta}^{fail}$ and that both Δ and $\Delta + \delta$ belong to $(\Delta_{Syst}, \overline{\Delta}^{fail})$. It thus follows that $\lambda^*(\Delta, \frac{1}{\Phi}) = \lambda^*(\Delta + \delta, \frac{1}{\Phi}) = 1$ so that $\mathcal{I}(.) = 0$. When the initial

[7] In the same way, it would be possible to define a similar indicator when the market impact increases for a constant shock.

shock is large enough and/or the price impact is important, most of the banks are already insolvent for the initial shock so that one naturally expects no amplification effect in such a case.

4 Empirical analysis : Banks participating to the 2015 CCAR stress tests

Stress testing is the analysis of how a generic entity (or object) such as a human body, a car, a bank... or a system of interacting entities such as a physical, biological or a financial system copes under pressure. For the specific case of a banking system, composed with interacting banks, there are actually various ways to design a stress test to assess its resilience. The stress test can be done on a micro prudential basis (bank by bank) or on a macro prudential basis, on a forward looking basis (based on projections of revenues, losses, capital, RWA in a given scenario) or on a point in time basis, on a specific asset class of assets (banking book or trading book) or on all the asset classes etc... While supervisory stress tests were first coordinated in the U.S. right after the events of 2008, as recalled in [Dent et al., 2016], internal stress tests have been conducted by banks themselves for risk management purpose since the early 1990s in order for the bank to (better) assess their trading portfolio's losses. This practice of stress tests was actually formalized in 1996 in line with the market risk amendment to Basel accords (see for instance [Dimson and Marsh, 1997]). Since the last few years, American and European banks are required to follow the guidelines of supervisors to conduct their stress tests. These *supervisory stress tests* (see [Hirtle and Lehnert, 2015] for a recent overview) are now described and sometimes criticized in various recent academic papers such as [Acharya et al., 2014], [Borio et al., 2014], [Flannery et al., 2017], [Greenlaw et al., 2012].

4.1 Comprehensive Capital Analysis and Review (CCAR) 2015

In their public document entitled CCAR 2015 summary instructions and guidance[8], the Board of Governors of the Federal Reserve System reports that the annual CCAR is "an intensive assessment of the capital adequacy of large, complex U.S. bank holding companies (BHC) and of the practices these BHC use to assess their capital needs". Each bank with consolidated assets of \$50 billion or more is required to participate and there is a total of 31 banks participating. As expected, one finds the well-known active international banks (identified as GSIBs) such as JP Morgan, Citigroup or Bank of America with consolidated assets higher than \$1800 billion. But one also finds smaller banks such as Comerica Incorporated, Discover Financial Services or Zions Bancorporation with consolidated assets lower than \$100 billion. Overall, the document provides the general instructions, qualitative and quantitative, required to perform the stress test. For instance, in section 3 entitled *Stress tests conducted by BHCs*, the document reports that each bank must conduct its stress test using five scenarios, three supervisory scenarios ranked by their severity (baseline, adverse, severely

[8] See [Federal-Reserve, 2015b].

adverse) and two BHC-defined scenarios also ranked by their severity (baseline, stress). One also learns from the document that the stress test is forward looking, from fourth quarter of 2014 to fourth quarter of 2016 so that each BHC is required, as of September 30, 2014, to estimate quantities such as revenues, losses, capital. Few BHCs (the six largest), with large trading operations, will "be required to include a global market shock as part of their supervisory adverse and severely adverse scenarios". The interesting feature of this global market shock is that it is an exogenous loss in the trading book that may reduce the (total) capital of the BHC, indeed Tier 1 capital. An important feature of the regulatory stress tests is that the BHC is not assumed to decrease its portfolio positions or RWAs due to losses from that global shock (static balance sheet assumption) and this is the point that our approach disputes. Within our framework, we make the assumption, as in the regulatory stress tests, that there is a point in time shock in the trading book. However, we explicitly allow the bank to react. Right after a systemic shock, within our model, the unique possibility for the bank to react is to sell a portion of its assets in order to comply with the regulatory capital requirement and we quantify the resulted price-mediated contagion. In our view, the main interest of the regulatory stress tests (American or not) as a macroprudential instrument is precisely to be able to anticipate the contagion process that results from banks reactions, i.e., the vicious circle between fire sales, price decrease and decrease in capital. But if banks are not allowed to react, the stress test is unable to quantify this source of indirect contagion. But then, as does [Goldstein, 2017] in his recent insightful book on stress testing and bank-capital reform, one can naturally question the usefulness of these regulatory stress tests.

4.2 Calibration of the model and equilibrium computation

For all banks involved in the 2015 CCAR, we collected from Bloomberg the total equity, that is Tier 1 + Tier 2 equity, the risk-weighted assets and total assets as of fiscal year end 2014. We display our data in Table 7 in the appendix. It is important to note that all these data can also be retrieved from the public annual report of each bank, available on their website. For the empirical analysis, we make the assumption that the implied aggregate risk weight for each bank is given by equation (8).

Within our model, the total equity Tier 1 plus Tier 2 corresponds to $E_{i,t}$ and the risk-weighted assets to $\alpha_i q_i P_t$. Moreover, the minimum ratio for total equity over risk-weighted assets is $\theta_{min} = 8\%$. Following equations (8), (14) and (15), it is straightforward to calibrate the values of α, Δ^{sale} and Δ^{fail} for each bank. We display these calibration results in Table (1). For example, the large international bank JP Morgan Chase has total equity of \$206 billion, risk-weighted assets of \$1,619 billion and total assets of \$2,572 billion. We find an implied risk-weight $\alpha = 63\%$, a fire sale threshold $\Delta^{sale} = 3.15\%$ and an insolvency threshold $\Delta^{fail} = 8.03\%$. This means that if JP Morgan Chase's assets lose more than 8.03%, the bank will be insolvent. If assets lose more than 3.15%, but less than 8.03%, JP Morgan Chase will remain solvent but with a capital ratio below 8%. It will thus have to engage in fire sales to bring back its capital ratio above 8%. From Table (1), one

Implied risk weight and critical thresholds

Bank	α	Δ^{sale}	Δ^{fail}
Ally Financial Inc	0.8612	0.0485	0.1141
American Express Company	0.8378	0.0683	0.1307
Bank of America Corporation	0.5997	0.0303	0.0768
BB&T Corporation	0.7690	0.0564	0.1144
BBVA Compass Bancshares, Inc	0.7747	0.0398	0.0993
BMO Financial Corp	0.3787	0.0247	0.0542
Capital One Financial Corporation	0.7710	0.0584	0.1164
Citigroup Inc	0.7017	0.0357	0.0898
Citizens Financial Group Inc	0.7976	0.0668	0.1263
Comerica Incorporated	0.9867	0.0268	0.1036
Discover Financial Services	0.8751	0.0854	0.1494
Fifth Third Bancorp	0.8498	0.0577	0.1218
HSBC North America Holdings Inc	0.4631	0.0367	0.0724
Huntington Bancshares Incorporated	0.8217	0.0489	0.1114
JPMorgan Chase &Co	0.6295	0.0315	0.0803
KeyCorp	0.9070	0.0576	0.1260
M&T Bank Corporation	0.8002	0.0616	0.1217
Morgan Stanley	0.5689	0.0503	0.0935
MUFG Americas Holdings Corporation	0.8507	0.0615	0.1254
Northern Trust Corporation	0.5721	0.0421	0.0859
Regions Financial Corporation	0.8278	0.0641	0.1260
Santander Holdings USA, Inc	0.5635	0.0401	0.0833
State Street Corporation	0.3934	0.0350	0.0654
SunTrust Banks, Inc	0.8538	0.0414	0.1069
The Bank of New York Mellon	0.4361	0.0218	0.0559
The Goldman Sachs Group, Inc	0.6661	0.0559	0.1063
The PNC Financial Services Group, Inc	0.8154	0.0690	0.1298
U.S. Bancorp	0.7893	0.0472	0.1073
Wells Fargo & Company	0.7364	0.0589	0.1143
Zions Bancorporation	0.7995	0.0707	0.1301

Table 1: Calibration

can see that $\underline{\Delta}^{sale} = 2.18\%$ and corresponds to the sale threshold of The bank of New York Mellon. In the same vein, $\overline{\Delta}^{fail} = 14.94\%$ corresponds to the insolvency threshold of Discover Financial services.

Once the relevant quantities of our model have been calibrated, i.e., the implied weight and the two thresholds of each bank, we consider the smallest Nash equilibrium under different scenarios (shock/price impact). In practice, while the market depth can be estimated using for instance daily market data, e.g., [Kyle and Obizhaeva, 2016], we here consider a set of possible market depths, ranging from very low to large ones. In what follows, the price impact is measured by $\frac{Q_{tot}}{\Phi}$.

To explicitly compute the Nash equilibrium, we make use of a fixed-point algorithm. Define the $[0,1]^p$- valued sequence $x^{(n)}$ by $x^{(0)} = (0,...,0)$ and $x^{(n+1)} = f(x^{(n)})$ where f is defined in equation (36). As f is non-decreasing and $x^{(0)} = (0,...,0)$, it is well-known that the sequence $x^{(n)}$ converges to the smallest fixed point of f, that is, the smallest Nash Equilibrium. There are actually various algorithms to find Nash equilibria in the literature. For instance, in [Echenique, 2007] (see also [Bigi et al., 2013]) the author presents an algorithm that find *all* the pure strategy equilibria for games with strategic complementarities, that is, the sequence $x^{(n)}$ converges to a Nash equilibrium x^* when n goes to infinity. Note that the first step of our algorithm $x^{(1)} = f(x^{(0)}) = f(0,...,0)$ corresponds to the liquidation vector in a myopic setting, that is, if banks calculate their optimal rebalancing strategy without taking into account the feedback from other banks, and which is used for example in [Greenwood et al., 2015].

It is important to point out that our approach is based on calibration and not on econometrics (see [Bajari et al., 2013] for an overview of game theory and econometrics). The parameters are not estimated statistically but are rather implied (or calibrated) from the data contained in the annual reports of banks using our model. It is thus similar in the spirit to the way an implied volatility is computed from the observed price using the Black and Scholes model or to the way an implied default probability is computed from the observed CDS spread using an intensity model. The main advantage of our approach is that it is transparent and simple to reproduce.

4.3 Empirical results

We now examine empirically the impact of an exogenous shock on this banking system made of 30 US bank holding companies. We shall first consider liquidation and convexity effects and we shall then discuss contagion.

Liquidation and convexity effects. Let us first discuss the case where asset market depth/liquidity is infinite, that is $\Phi = \infty$ or, equivalently, $\frac{Q_{tot}}{\Phi} = 0$. In figure 4.3, the blue line displays the dollar size of fire sales in the banking system (Y axis) as a function of the shock size Δ (X axis). As expected, when the shock Δ on risky assets is such that $\Delta \leq \underline{\Delta}^{sale} = 2.18\%$, all banks remain solvent and with a regulatory capital ratio above 8% and there is no need to liquidate assets. Symmetrically, when $\Delta \geq \overline{\Delta}^{fail} = 14.94\%$, all banks become insolvent due to the size of the shock and are fully liquidated. Between those two thresholds, as opposed to the case of a single bank in which fire sales increase linearly with Δ, we observe a *non-linear relationship* between fire sales and shock size in the case of a system with multiple banks. The volumes liquidated turn out to be highly convex for shock slightly above $\underline{\Delta}^{sale} = 2.18\%$. For instance, the volume of fire sales when $\Delta = 6\%$ is equal to \$7,103 billion, which is much more than twice the volume of fire sales when $\Delta = \frac{6\%}{2} = 3\%$, which is equal to \$1,957 billion. This is due to the fact that the larger the shock, not only the larger the volumes liquidated by a given bank, but also the greater the number of banks engaging in fire sales.

From a regulatory perspective, our model enables to estimate such convexity effects and, more generally, the endogenous reaction of the banking system to an exogenous shock and hence anticipate potential destabilizing loops in a macro-prudential context. In order to avoid too large deleveraging phenomenon, the regulator could temporarily decrease capital requirements for banks after a large shock, that is decrease θ_{min}. Our approach provides a simple theoretical framework together with numerical results that enable a regulator to assess the capital requirement relief needed for a given constraint. Consider once again the no price impact case and let us assume that the regulator would like (arbitrarily) to limit the volume of fire sales when $\Delta = 6\%$ to \$6,000 billion. We know that if $\theta_{min} = 8\%$, liquidations will amount to \$7,103 billion. We find numerically that the regulator should lower its capital requirement to $\theta_{min} = 6.75\%$ in order to constrain the fire sales volume to \$6,000 billion when $\Delta = 6\%$. A similar analysis can be obviously done when the price impact is positive.

Consider now the case in which the price impact is positive. Figure 4.3 displays the volume of fire sales in the banking system as a function of Δ for different values of $\frac{Q_{tot}}{\Phi}$. As expected, we find that the greater the size of the banking system compared to asset market depth, the greater the volume of fire sales following a given exogenous shock Δ. Quite interestingly, one can see from figure 4.3 that the excess fire sales due to market frictions are *maximal for intermediate shocks*. For instance, for an exogenous shock $\Delta = 6\%$, the volume of fire sales when $\frac{Q_{tot}}{\Phi} = 3\%$ is doubled compared to the case without price impact. When banks have large positions compared to asset liquidity, market frictions are so important that any bank that starts deleveraging its portfolio is going to have a very large impact on asset prices and lead to a contagion of defaults (see green line in figure 4.3). This extreme behavior is actually caused within our model by the fact that liquidations all take place at the same date $t+1$ whatever their size and price impact. As already said, in practice, to avoid this kind of problem in which each bank liquidates the same asset at the same time, regulators may temporarily decrease the required RBC.

Amplification effect. In table 2, we compute from table 8 (see the appendix) the possible amplification due to a marginal shock (see equation 46). It is for an intermediate shock size and price impact that the amplification effect is the highest. Consider once again an initial shock $\Delta = 6\%$ and assume that its size is increased by 100 basis points. Without price impact, the fraction of insolvent banks increases from 6.6% to 10% (see table 8). With a price impact of 5%, this fraction

$$\mathcal{I}(\Delta, \delta = 100bps, \frac{Q_{tot}}{\Phi})$$

Δ_1 to Δ_2 || $\frac{Q_{tot}}{\Phi}$	1%	3%	5%	6.75%	8.5%
5% to 6%	1	1.5	1.5	3	3
6% to 7%	2	3	7	7	0
7% to 8%	3.8	2.5	1.5	0	0.5
8% to 9%	0.5	1.25	1	0.25	0
9% to 10%	2.5	4	0	0	0
10% to 11%	1.25	0.25	0.25	0	0
11% to 12%	1.2	0.2	0	0	0

Table 2: Amplification effect

increases now from 40% to 63.33%, which means that the fraction of insolvent banks is multiplied approximately by 7 i.e., $\mathcal{I} = \frac{0.633-0.4}{0.1-0.066} \approx 7$. Table 2 provides the different values of the index $\mathcal{I}$ in various scenarios and one can see that for moderate initial shocks and moderate price impacts, in general, $\mathcal{I}$ is greater than one. However, when the price impact is high (say higher than 7%) and when the initial shock is large (say greater than 9%), there is no amplification effect because most of the banks are already insolvent from the initial shock. In such a case, when the size of the shock is increased by 100 bps, this marginal shock has virtually no consequence and this explains why $\mathcal{I}$ is close to zero or even equal to zero.

4.4 GSIB capital surcharge

In this section, we examine what would happen if GSIB capital surcharges as of 2016 were already fully implemented in 2014.

On macroprudential regulation. The purpose of macroprudential regulation is to limit the likelihood and costs of contagion [Greenlaw et al., 2012] (see also the policy paper of [Clerc et al., 2016]). In the academic literature, various macroprudential tools such as time varying capital, contingent capital or higher quality capital have indeed been considered [Hanson et al., 2011]. Few years ago, the Basel Committee proposed a methodology to classify a bank as a global systemically important bank (GSIB) or not. This methodology is based on a final score supposed to reflect the *systemic impact* of the bank, and is computed as a function of indicators such as the size and interconnexions of the bank ([Board, 2017]). A bank classified as a GSIB will be required, depending upon its final score, to have a higher loss absorbency (HLA), or capital surcharge, expressed in percentage of the RWA, that is, GSIBs are required to hold additional capital. The final score is expressed in basis points and the HLA is a piecewise constant function; as long as the final scores falls between two thresholds, typically x and $x + 100$ bps, the HLA of the bank is constant and is equal to one of the five buckets 1%, 1.5%, 2.%, 2.5%, 3.5% (in 2017, the top bucket 3.5% is still empty). This capital

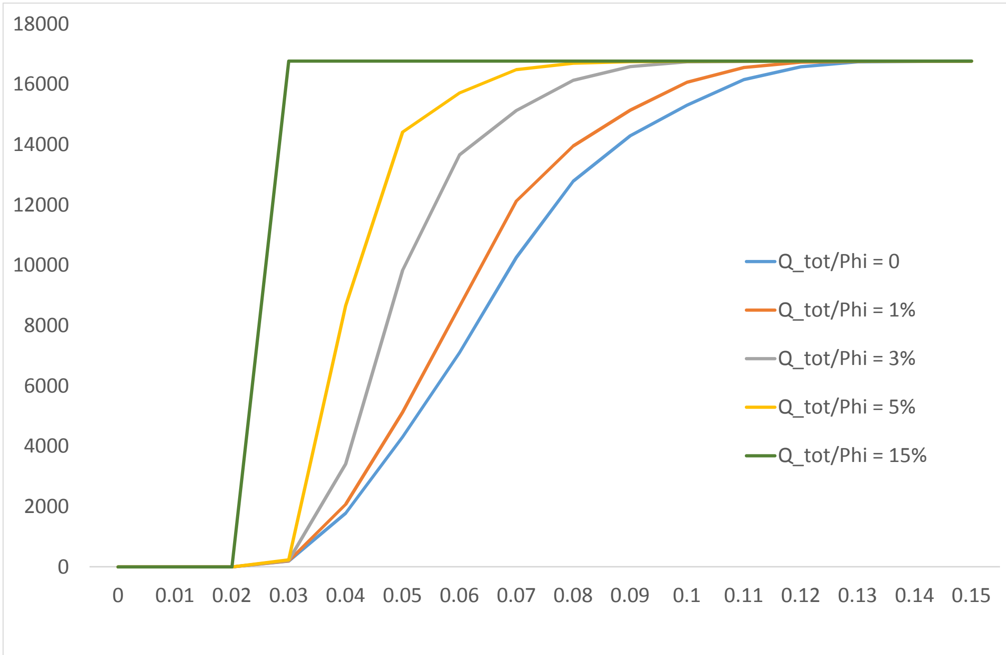

Figure 2: Volume of fire sales in Bn dollars in the banking system (Y axis) as a function of shock size Δ (X axis)

surcharge requirement began in January 2016 and will be fully implemented in 2019. The 2016 list of American banks classified as GSIBs can be found in a document published by the Financial Stability Board[9].

The situation as of 2014. Consider a shock $\Delta = 6\%$. From table 1, only two banks, namely BMO Financial corp and The Bank of New York Mellon, have an insolvency threshold Δ^{fail} lower than 6%. When there is no price impact, i.e., $\frac{Q_{tot}}{\Phi} = 0\%$, only these two banks fail. Since there are 30 banks, the fraction of insolvent banks thus is equal to 6.66%, which is the number reported in table 8 (see the appendix) for this initial shock of 6%. When the price impact is now equal to 3%, i.e., $\frac{Q_{tot}}{\Phi} = 3\%$, seven banks are actually insolvent so that the fraction of insolvent banks is equal to 23.3%. Among them, we find the well-known large international financial institutions Bank of America, HSBC North America Holding, JP Morgan Chase but we also find, beside BMO Financial corp and The Bank of New York Mellon, banks such as Santander Holding USA and State Street

[9]See the website of the FSB, http://www.fsb.org/wp-content/uploads/2016-list-of-global-systemically-important-banks-G-SIBs.pdf

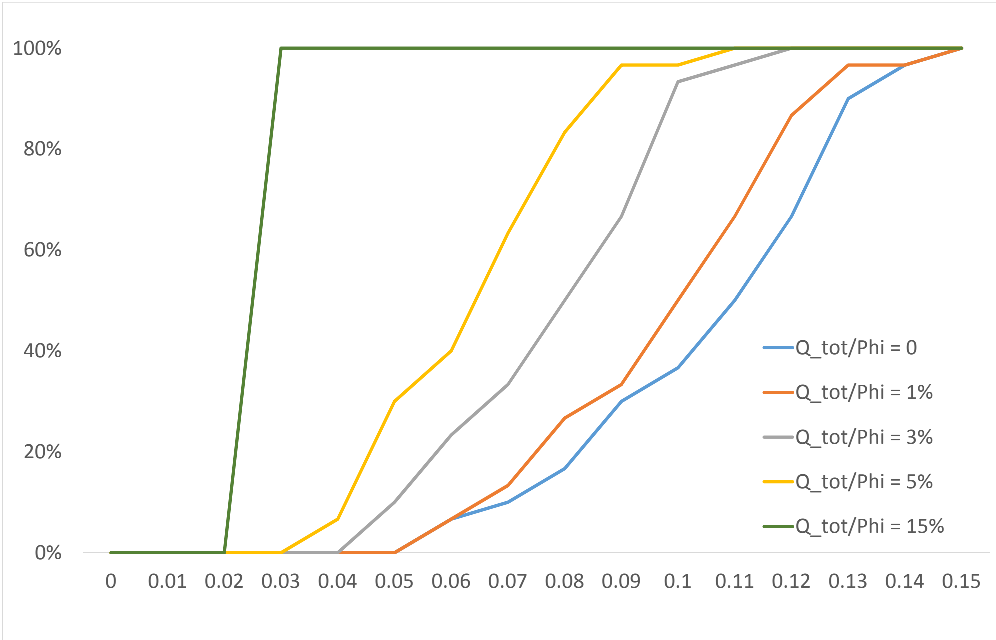

Figure 3: Proportion of insolvent banks (Y axis) as a function of shock size Δ (X axis)

Corporation, whose total assets are much lower (respectively equal to 118 and 274 billion of dollars). It is interesting to note that out of the seven banks that are insolvent under our scenario (initial shock of 6%, price impact of 3%), six are indeed classified as GSIB. The subset of insolvent banks in our model thus is a good predictor of the subset of banks identified as GSIBs, including banks with a total value of the assets much lower than the well-known large international banks.

The situation of 2014 assuming the 2016 GSIB capital surcharge. It is now interesting to conduct our exercise: as a function of the price impact, what can be said of the fraction of insolvent banks at equilibrium if the capital surcharge would have been fully implemented in 2014 ? From the information contained in the 2016 list, the capital surcharge as a function of the RWA is equal to 2.5% for Citigroup and JP Morgan Chase, 2% for Bank of America and HSBC, 1.5% for Goldman Sachs and Wells Fargo, 1% for Bank of New York Mellon, Morgan Stanley, Santander Holding USA and State Street. To be very concrete, as of 2014, the RWA of Bank of America is equal to $1,262 Billion (see table 7) and is allocated to the bucket 2%. As a result, the additional buffer is equal to $25 Billion so that the new total capital is equal to $187 Billion instead of $162 Billion.

When the initial shock is equal to 6% and the price impact equal to 3%, beside BMO Financial corp and The Bank of New York Mellon, only State Street Corporation defaults, which means that only three banks are now insolvent, that is, 10% of the banks are insolvent. Quite interestingly, the large international banks are able to survive at equilibrium. Compared to the no capital buffer case, the difference is important as the fraction of insolvent banks is reduced by more than 50%, i.e., from 23% to 10%.

5 Fine tuning the price-mediated contagion framework : theory and empirical analysis

Up to now we have made the assumption that each bank is exposed to a single marketable asset subject to market risk (and possibly to counterparty risk). As a result, the model can be easily calibrated as the unique regulatory risk weight is implied from the ratio of the risk-weighted assets and total assets. This single risky asset actually provides an upper bound to the fire sale problem since in practice, many banks are also exposed to loans, which are risky non marketable assets that are difficult to sell in the short-run.

We now consider a model in which a given bank is exposed not only to a marketable asset subject to market risk but also to a non-marketable asset (loans) subject to credit risk. As opposed to the marketable asset, which is marked to market, the value of the loans is not. Instead, the value reported in the balance sheet is derived from a discounted cash-flow model or an amortized cost method and may not coincide with its resale value. From the well-known adverse selection problem ([Diamond and Rajan, 2011]), loans are highly illiquid and their resale value is typically low (at least in the short-run). We shall make here the assumption that, when considering a short-term horizon to restore its risk-based capital ratio, the bank is unable to find a buyer so that the resale value of the loans is equal to zero. A bank for which its risk-based capital ratio is lower than the minimum required after the adverse shock thus can only sell a fraction of its marketable assets.

5.1 The two risky assets model

In the one risky asset model, the total value of the assets is equal at time t to $A_{i,t} = v_i + P_t q_i$, where v_i is the cash and q_i is the quantity of the risky marketable asset held by bank i. In the two risky assets model, each bank i has now a set of loans (*banking book*) whose value at time t is denoted $V_{i,t}^{\text{Bank}}$, and a marketable asset (*trading book*) whose value is equal $V_{i,t}^{\text{Trad}} = q_i' P_t$, where q_i' is the quantity of the risky marketable asset held by the bank in the two-assets model[10]. The total value of the assets thus is equal to

$$A_{i,t} = v_i + V_{i,t}^{\text{Bank}} + V_{i,t}^{\text{Trad}} \tag{47}$$

The risk-weighted assets thus is equal to

[10] We typically add a prime to each quantity to indicate that it is computed in the two risky assets model.

$$\mathrm{RWA}_{i,t} = \mathrm{RWA}_{i,t}^{\mathrm{Bank}} + \mathrm{RWA}_{i,t}^{\mathrm{Trad}} \tag{48}$$

and is defined as the sum of the risk-weighted assets of the banking book ($\mathrm{RWA}_{i,t}^{\mathrm{Bank}}$) and the risk-weighted assets of the trading book ($\mathrm{RWA}^{\mathrm{Trad}}$), where

$$\mathrm{RWA}_{i,t}^{\mathrm{Trad}} = \alpha_i \, V_{i,t}^{\mathrm{Trad}} \tag{49}$$

$$\mathrm{RWA}_{i,t}^{\mathrm{Bank}} = \beta_i \, V_{i,t}^{\mathrm{Bank}} \tag{50}$$

In this framework, α_i is the (average) risk weight of the trading book while β_i is the (average) risk weight of the banking book. The risk-based capital ratio of each bank i at time t is equal to

$$\theta'_{i,t} := \frac{E'_{i,t}}{\mathrm{RWA}_{i,t}} = \frac{A_{i,t} - D_i}{\mathrm{RWA}_{i,t}^{\mathrm{Trad}} + \mathrm{RWA}_{i,t}^{\mathrm{Bank}}} = \frac{A_{i,t} - D_i}{\alpha_i V_{i,t}^{\mathrm{Trad}} + \beta_i V_{i,t}^{\mathrm{Bank}}} > \theta_{min} \tag{51}$$

and is (as in the one asset model) assumed to be greater than the minimum required. Since the total value of the assets $A_{i,t}$ but also the total debt D_i coincide in both models, it thus follows that

$$E_{i,t} = E'_{i,t} \tag{52}$$

Assume now that that at time t^+, the price of the marketable asset is hit by a shock Δ so that its price at time t^+ is equal to $P_t(1-\Delta)$. Noting that the value of the cash v_i as well as the value of debt D_i are not impacted by the shock Δ, at time t^+, the balance-sheet of bank i is given below.

Balance-sheet at time t^+ before deleveraging

Assets	Liabilities
Cash: v_i Banking book $V_{i,t}^{\mathrm{Bank}}$ Trading book $V_{i,t^+}^{\mathrm{Trad}} = q'_i P_t(1-\Delta)$	Debt: D_i Equity: E_{i,t^+}
A_{i,t^+}	$E_{i,t^+} + D_i$

The risk-based capital ratio of the bank i at time t^+ thus is equal to

$$\theta'_{i,t^+}(\Delta) = \frac{\max\{E_{i,t} - q'_i P_t \Delta; 0\}}{\alpha_i q'_i P_t(1-\Delta) + \beta_i V_{i,t}^{\mathrm{Bank}}} \tag{53}$$

and, when positive, is a decreasing function of the shock size.

Lemma 5 *Each bank $i \in B$ is characterized by the two following critical thresholds.*

$$\Delta_i'^{sale} := \frac{\Delta_i'^{fail} - \theta_{min}\left(\alpha_i + \frac{\beta_i V_{i,t}^{\mathrm{Bank}}}{q'_i P_t}\right)}{1 - \alpha_i \theta_{min}} > 0 \tag{54}$$

$$\Delta_i'^{fail} := \frac{E'_{i,t}}{q'_i P_t} > 0 \tag{55}$$

with $\Delta_i'^{sale} < \Delta_i'^{fail}$

Let $x'_i \in [0,1]$ be the fraction of the marketable asset sold in the two assets model. After deleveraging, the balance sheet is as follows.

Balance-sheet of bank i at date $t+1$ after deleveraging

Assets	Liabilities
Cash: $v_i + x'_i q'_i P_{t+1}(\boldsymbol{x}', \Phi)$	Debt: D_i
Banking book : $V_{i,t}^{\text{Bank}}$	
Trading book: $(1-x'_i) q'_i P_{t+1}(\boldsymbol{x}', \Phi)$	Equity: $E'_{i,t+1}$
$A_{i,t+1} = v_i + V_{i,t}^{\text{Bank}} + q'_i P_{t+1}(\boldsymbol{x}', \Phi)$	$E'_{i,t+1} + D_i$

where $P_{t+1}(\boldsymbol{x}', \Phi)$ is given as before by equation (19) (see assumption 3). The regulatory capital ratio of bank i at time $t+1$, after deleveraging, is equal to

$$\theta'_{i,t+1}(\boldsymbol{x}', \Delta) = \frac{E_{i,t+1}(\boldsymbol{x}')}{\alpha_i q_i P_{t+1}(\boldsymbol{x}', \Phi)(1-x'_i) + \beta_i V_{i,t}^{\text{Bank}}} \tag{56}$$

where, as before, we make the natural convention that $\theta_{i,t+1}(\boldsymbol{x}, \Delta) = 0$ when $x_i = 1$ and $E_{i,t+1} = 0$.

From a pure mathematical point of view, although the risk-based capital ratio in equation (23) is not equal to equation (56), nothing is changed regarding the conditions of existence (and uniqueness) of a Nash equilibrium.

From a financial point of view, under the assumption that the banking book can not be liquidated, there is however an important difference between the one risky asset model and the two risky assets model in the no price impact case. In the first one, as long as bank i is still solvent after the shock, it is always able to restore its regulatory capital ratio back above the minimum required because the risk-weighted assets tends to zero when the fraction of the risky asset sold tends to one. This is however not anymore true in the two risky assets model. Assume as before that $E'_{i,t^+} = E_{i,t} - V_{i,t}^{\text{Trad}}\Delta > 0$ and recall that $V_{i,t}^{\text{Trad}} = q'_i P_t$. When the bank sells a fraction $x'_i > 0$, the RBC is equal to

$$\theta'_{i,t^+}(\Delta, x'_i) = \frac{E_{i,t} - V_{i,t}^{\text{Trad}}\Delta}{\alpha_i q'_i (1-x'_i) P_t (1-\Delta) + \beta_i V_{i,t}^{\text{Bank}}} \tag{57}$$

In the limiting case in which x'_i tends to one, the RBC tends to $\frac{E_{i,t} - V_{i,t}^{\text{Trad}}\Delta}{\beta_i V_{i,t}^{\text{Bank}}} < \infty$ and this limit might be lower than θ_{min}. It is actually the case when the banking book is large enough compared to the trading book. In such a case, when the bank sells 100% of its trading book, since most of the RWA is due to credit risk, the RWA after deleveraging remains close to the RWA before the deleveraging, so that the risk-based capital ratio (after deleveraging) remains very close to the risk-based capital ratio (before deleveraging), lower than θ_{min}. One can thus compute a *critical shock size* denoted Δ^{crit} such that $\frac{E_{i,t} - V_{i,t}^{\text{Trad}}\Delta^{crit}}{\beta_i V_{i,t}^{\text{Bank}}} = \theta_{min}$ and such that any shock Δ above Δ^{crit} will leave the bank unable to restore its capital ratio above θ_{min} even if it solvent and liquidates its

Severely adverse scenario

Bank holding company	Actual Q3	Proj. min. 2015-2016
Bank of America	15.8%	10.7%
Citigroup	17.7%	9.4%
The Goldman Sachs	19.8%	7.6%
JP Morgan Chase & Co	15%	8.36%
Morgan Stanley	19.8%	7.4%
Wells Fargo & Company	15.6%	10.5%
Average	17.26%	9%

Table 3: Banks' capital ratios

trading book in full. It is easy to show that this critical shock is equal to

$$\Delta'^{crit} = \frac{E_{i,t} - \theta_{min}\beta_i V_{i,t}^{\text{Bank}}}{V_{i,t}^{\text{Trad}}} \tag{58}$$

5.2 Empirical analysis : the six American banks with significant trading operations

Benchmark scenario As part of the regulatory stress-test, in addition to the other scenarios (adverse and severely adverse), the six American BHCs with significant trading operations, namely Bank of America Corporation, Citigroup inc, The Goldman Sachs Group, Inc, JPMorgan Chase & Co, Morgan Stanley and Wells Fargo & Company, all classified as GSIBSs (although in a different bucket), are also required to assume a global market shock. As already seen, this shock is applied to BHCs' trading book (private-equity positions and counterparty exposures) as of a "point in time, resulting in instantaneous loss and reduction of capital" and is thus identical to the shock we assume in the trading book of each bank within our model. The overall loss for each bank in the severely adverse scenario is equal to the net income before taxes which is negative. This loss ranges from \$19 billion for Morgan Stanley to \$54.8 billion for JPMorgan[11]. We reproduce in table 3 the information found p 15 in the document CCAR 2015: Assessment Framework and Results (see [Federal-Reserve, 2015a]) for the six banks under consideration.

Two banks out of the thirty one, namely Goldman Sachs and Morgan Stanley, have a (minimum projected) risk-based capital ratio *lower* than the minimum required (i.e., 8%) in the severely adverse scenario. From table 3, before the stress, the average risk-based capital ratio is equal to 17.26% and this average falls to approximately 9% in the severely adverse scenario. In terms of risk-based capital ratio, the average loss is equal to approximately 8% which gives us a natural benchmark to set the initial shock Δ.

[11] See [Federal-Reserve, 2015c]

Implied risk weights and critical thresholds

BHC||Quantities	α	β	Δ^{sale}	Δ^{crit}	Δ^{fail}
Bank of America	0.494	0.8460	0.168	0.201	0.369
Citigroup	0.341	0.898	0.1068	0.131	0.27718
The Goldman Sachs	0.708	0.722	0.101	0.152	0.1919
JP Morgan Chase& Co	0.365	0.773	0.0926	0.11913	0.2409
Morgan Stanley	0.4737	0.7211	0.0928	0.12726	0.174
Wells Fargo & Company	0.366	0.85	0.270	0.291	0.542

Table 4: Parameters of the two risky assets model

Risk-based capital ratio after the shock but before liquidations

Bank holding company	Actual Q4	$\Delta = 5\%$	$\Delta = 9\%$	$\Delta = 10\%$	$\Delta = 11\%$	$\Delta = 12\%$
Bank of America	14.2%	12.4%	11%	10.6%	10.2%	9.8%
Citigroup	12.8%	10.6%	8.8%	8.3%	7.9%	7.4%
The Goldman Sachs	15.9%	12.2%	8.9%	8.1%	7.3%	6.4%
JP Morgan Chase& Co	12.8%	10.2%	8.1%	7.6%	7.1%	6.6%
Morgan Stanley	16.4%	12%	11%	7.3%	6.4%	5.4%
Wells Fargo & Company	15.5%	14.1%	13%	12.8%	12.5%	12.2%
Average	14.6%	11.9%	10.1 %	9.1 %	8.56 %	7.9 %

Table 5: Situation before liquidation

Model calibration and shocks assumption. We explain in the appendix the methodology used to derive from a bank's annual report its balance sheet split by trading and banking books (V^{Trad} and V^{Bank}) and their associated risk-weighted assets (RWA^{Trad} and RWA^{Bank}). Once these quantities are calibrated, the two weights α and β are easy to obtain. We report in table 4 these two implied regulatory risk weights and the three critical thresholds for each bank.

Within our model, as in the one asset model, we start with risk-based capital ratios 2014:Q4 (fully loaded) and we obtain table 5 as a function of the severity of the shock Δ before the possible bank's reaction. The various risk-based capital ratios thus do not depend upon the price impact (or market liquidity) measured by $\frac{Q_{tot}}{\Phi}$.

From table 5, one can clearly see that as long as the shock is lower than 9%, the risk-based capital ratio remains higher than the required minimum for each bank so that there is no liquidation, i.e., no fire sale. However, when the shock is equal to 10%, the average risk-based capital ratio falls to 9.1% and two banks, namely JPMorgan and Morgan Stanley have a risk-based capital ratio *lower* than the required minimum. Interestingly, a shock of 10% within our framework yields a result identical to the CCAR in the severely adverse scenario. In both cases, two banks are under

Risk-based capital ratio after liquidations at equilibrium ($\frac{Q}{\Phi} = 5\%$)

Bank holding company	Actual Q4	$\Delta = 10\%$	$\Delta = 11\%$	$\Delta = 12\%$
Bank of America	14.2%	9.53%	9.06%	8.64%
Citigroup	12.8%	8%	7.5% (insolv.)	6.88% (insolv.)
The Goldman Sachs	15.9%	8%	8%	8%
JP Morgan Chase& Co	12.8%	7.41% (insolv.)	6.6% (insolv.)	5.87% (insolv.)
Morgan Stanley	16.4%	7.85% (insolv.)	5.74% (insolv.)	3.85% (insolv.)
Wells Fargo & Company	15.5%	12%	11.64%	11.39%

Table 6: Situation after (possible) liquidation : equilibrium

capitalized (i.e., with a risk-based capital ratio lower than the required minimum) and the average stressed risk-based capital ratio is equal to 9%. Within our model, the average loss in terms of risk-based capital ratio is equal to 550 bps and thus is much lower than the 800 bps found in the supervisory stress test. Table 6 shows the risk-based capital ratio at equilibrium, that is, after the liquidation process, when the price impact is set to 5%.

In the two risky assets model, due to the existence of the banking book which can not be liquidated, the risk-weighted asset of a given bank i does not tend to zero when x_i tends to one. As a result, and contrary to the one asset model, it may be the case that at equilibrium, the capital ratio of an insolvent bank *remains positive but lower than 8%.*

Contagion of failures. In an early survey on the subject, [Pericoli and Sbracia, 2003] offer five possible definitions of financial contagion. Some of them involve spillover effects of volatility or significant comovements of prices[12] but unfortunately, no definition turns out to be suitable for our stress test model of the banking system.

Following [Elliott et al., 2014] (see also [Acemoglu et al., 2015]), it is more appropriate to define contagion in terms of cascade of failures after an exogenous shock. Since we work with a static model, as in [Allen and Gale, 2000], contagion should rather be modeled as an *equilibrium phenomenon.*

Contagion of failures is said to occur when a subset of banks have their capital ratio greater than the required minimum 8% *after* the shock but are insolvent *at equilibrium*, i.e., after the liquidation process. Formally, such a contagion of failures is said to occur if there exists at least one bank $i \in B$ such that $\theta_{i,t^+}(\Delta) > \theta_{min}$ and $\theta_{i,t+1}(\boldsymbol{x}^*, \Delta) < \theta_{min}$. Note however that this definition does not rule out the existence of a bank $j \neq i$ such that $\theta_{j,t^+}(\Delta) < \theta_{min}$ but such that $\theta_{j,t+1}(\boldsymbol{x}^*, \Delta) \geq \theta_{min}$. The contagion of failures is made possible within our framework due to the existence of a positive price impact when banks deleverage. Without positive price impact and/or without deleveraging, contagion of failures makes no sense.

[12]See [Jayech, 2016] for a recent paper in which the authors investigate whether there is a contagion phenomenon between the stock markets during the July-August-2011 stock market crash.

Note importantly that such a contagion of failures can not be captured by the supervisory stress tests since they do not take into account bank deleveraging. This contagion of failures can only be partially captured by non-equilibrium fire sales models (e.g., [Greenwood et al., 2015]) since some banks may be solvent after the first round of deleveraging but might be insolvent at equilibrium.

Consider a shock equal to 10%. From table 5, we know that before any liquidation, only two banks are undercapitalized but solvent, JPMorgan and Morgan Stanley. Note that Goldman Sachs and Citigroup have a risk-based capital ratio slightly above 8% so that they do not need to react. However, since JPMorgan and Morgan Stanley must react (i.e., they have to sell a fraction of their marketable assets to try to restore their risk-based capital ratio back above the required minimum), these liquidations will impact the price of the marketable asset. Depending on the price impact, *at equilibrium*, Goldman Sachs and Citigroup may also have to react. For Citigroup, when the price impact (measured by $\frac{Q_{tot}}{\Phi}$) is lower than 4%, no reaction is needed. However, when the price impact is equal to 5%, to restore back its risk-based capital ratio, Citigroup is forced to sell an important fraction of its holding, actually 87%. When the price impact is 6%, Citigroup is now unable to restore its capital ratio and thus is insolvent, it must liquidate 100% of its holding. For Goldman Sachs, as long as the price impact is lower than 2%, no reaction is required. As for Citigroup, when the price impact is equal to 5%, the fraction of the marketable asset liquidated to restore back its risk-based capital ratio is rather high (51%) and when the price impact is equal to 8%, Goldman Sachs must sell almost 100% of its trading book (98.5%). When the price impact is equal to 9%, Goldman Sachs is insolvent.

This analysis shows that price impact and, more generally, market liquidity are crucial ingredients for understanding contagion (at equilibrium) in a severe scenario. As we have seen, when the price impact is low enough, typically lower than 5%, there is no contagion of failures since both Citigroup and Goldman Sachs remain solvent at equilibrium (for very low price impact, even without any liquidation at all). However, when this price impact is equal to 10%, Citigroup and Goldman Sachs are unable to restore their capital ratio back above the minimum at equilibrium. In such a scenario, the contagion of failures is important since two banks out of six (33%) fail at equilibrium because of a pure price-mediated contagion effect. When one now considers a price impact equal to 15%, Bank of America is also insolvent so that the contagion of failures increases up to 50%. It is interesting to note that even when the price impact is equal to 15%, Wells fargo does not even need to react. Our analysis suggests that Wells Fargo is by far the most resilient bank.

For such high price impact, price mediated contagion is a real issue which is unfortunately not captured by supervisory stress tests. Our results could however be used to design the capital surcharge against the contagion of failures generated by large fire sales.

6 Conclusion

In this paper, we developed a stylized framework of strategic macro stress test in which banks are hit by a common shock and may have to sell assets. This naturally leads to a game which strategic complementarities for which we showed the existence of a Nash equilibrium and the way to choose it when non-unique. We then explained how to calibrate the parameters of our model to public data and studied empirically the set of American banks that were part of the regulatory stress as of 2015 (CCAR), providing various comparative analyses as a function of shock size and/or asset liquidity. Finally we explained how our framework can be used to draw macro prudential regulatory measures such as the capital surcharge for GSIBs.

7 Appendix A: proofs

Proof of lemma 1

When $\Delta > \frac{E_{i,t}}{q_i P_t}$, then

$$\theta_{i,t^+}(\Delta) = 0 \tag{59}$$

When $0 \leq \Delta \leq \frac{E_{i,t}}{q_i P_t}$, then

$$\theta_{i,t^+}(\Delta) = \frac{E_{i,t} - q_i P_t \Delta}{\alpha_i q_i P_t (1-\Delta)} \tag{60}$$

and

$$\theta'_{i,t^+}(\Delta) = \frac{\alpha_i q_i P_t (E_i - q_i P_t)}{(\alpha_i q_i P_t (1-\Delta))^2} \tag{61}$$

which is negative under assumption 2. As a consequence, $\theta_{i,t^+}(\Delta)$ is a decreasing function of Δ.

Proof of lemma 3

Part 1. We shall first prove the following Lemma:

Lemma A 1 *Assume that $\Delta < \Delta_i^{fail}$ and $v_1 \leq v_2$. Then, for all $x_i \in [0,1)$, $\theta_{i,t+1}(x_i, v_1, \Delta) \geq \theta_{i,t+1}(x_i, v_2, \Delta)$.*

Proof. Consider first the case in which, for a given $x_i < 1$, $\Delta_i^{fail} - \Delta(x_i, v_2) > 0$. Since $v_1 \leq v_2$ and since $\Delta(x_i, v)$ is an increasing function of v, it thus follows that $\Delta_i^{fail} - \Delta(x_i, v_1) > 0$. It is not difficult to show that if $v_1 \leq v_2$, then, $\theta_{i,t+1}(x_i, v_1, \Delta) \geq \theta_{i,t+1}(x_i, v_2, \Delta)$. Assume now that for a given $x_i < 1$, $\Delta_i^{fail} - \Delta(x_i, v_2) \leq 0$ so that $\theta_{i,t+1}(x_i, v_2) = 0$. By definition, capital ratios are positive and we have $\theta_{i,t+1}(x_i, v_1) \geq \theta_{i,t+1}(x_i, v_2)$ which concludes the proof of lemma 1 $\square$

We can now prove part 1 of lemma 3. When $BR_i(v_2, \Delta) = 1$, we clearly have $BR_i(v_1, \Delta) \leq BR_i(v_2, \Delta)$. Assume that $BR_i(v_2, \Delta) < 1$ and note that this implies that $\Delta < \Delta_i^{fail}$.

- In the case where $BR_i(v_2, \Delta) = 0$, then we have $\theta_{i,t+1}(0, v_2) \geq \theta_{min}$. Given lemma 1, this means that $\theta_{i,t+1}(0, v_1) \geq \theta_{min}$ and so $BR_i(v_1, \Delta) = 0 \leq BR_i(v_2, \Delta)$.

- In the case where $0 < BR_i(v_2, \Delta) < 1$, this implies that $\theta_{i,t+1}(BR_i(v_2, \Delta), v_2) = \theta_{min}$. Given lemma 1, this means that $\theta_{i,t+1}(BR_i(v_2, \Delta), v_1) \geq \theta_{min}$ which implies that $BR_i(v_1, \Delta) \leq BR_i(v_2, \Delta)$ and concludes the proof $\square$

Part 2.

The proof of part 2 is similar. One can show that for all $v > 0$, if $\Delta_1 \leq \Delta_2$, then, for all $x_i \in [0,1)$, $\theta_{i,t+1}(x_i, v, \Delta_1) \geq \theta_{i,t+1}(x_i, v, \Delta_2)$ which implies part 2 of lemma 3. $\square$

Proof of Proposition 1 For an arbitrary shock Δ and an arbitrary market depth $\Phi > 0$, from lemma 3, the best response $BR_i(x_{-i})$ of each bank $i = 1, ..., p$ is an increasing function of x_{-i}. By letting $f := \prod_{i=1}^{p} BR_i$, f is a non-decreasing function from X to X, where $X = [0,1]^p$ so that from Tarski's theorem, $\mathcal{F}_\Delta$ is not empty and is moreover a complete lattice and this concludes the proof of existence. Note that although the best response is unique in our model, the existence result would work for a best response correspondence.

The proof that the subsets S^* and D^*, when non empty, are composed with consecutive integers relies on the two following lemma.

Lemma A 2 *For all initial shocks $\Delta > 0$ and all equilibria $\boldsymbol{x}^* \in \mathcal{F}_\Delta$*

1. *if $x_i^* < 1$, then, $\Delta(\boldsymbol{x}^*) < \Delta_i^{fail}$.*
2. *if $x_i^* = 1$, then, $\Delta(\boldsymbol{x}^*) \geq \Delta_i^{fail}$.*

Proof Part 1. Consider a given equilibrium $\boldsymbol{x}^* \in \mathcal{F}_\Delta$. If $x_i^* < 1$, bank i is solvent, i.e., $\theta_{i,t+1}(x_i^*, x_{-i}^*) \geq \theta_{min}$, then, its total capital must be positive; $E_{i,t+1}(\boldsymbol{x}^*) > 0$. From the equivalence provided in equation (27), it thus must be the case that $\Delta(\boldsymbol{x}^*) < \Delta_i^{fail}$. Part 2. Assume that the contrary is true, i.e., when $\boldsymbol{x}^* := (x_i^* = 1, x_{-i}^*)$, $\Delta(\boldsymbol{x}^*) < \Delta_i^{fail}$. From equation (27), this means that the total capital $E_{i,t+1}(\boldsymbol{x}^*) > 0$. But then, since the total capital is positive for each $x_i \in [0,1]$, given $x_{-i}^* \in [0,1]^{p-1}$, there exists $x_i^* < 1$ such that $\theta_{i,t+1}(x_i^*, x_{-i}^*) = \theta_{min}$, and this contradicts the optimality of the best response $x_i^* := BR_i(x_{-i}^*) = 1$.$\square$

Lemma A 3 *For all initial shocks $\Delta > 0$ and all equilibria $\boldsymbol{x}^* \in \mathcal{F}_\Delta$*

1. *if there exists $0 \leq i_1 \leq p$ such that $x_{i_1}^* < 1$, then $x_i^* < 1$ for all $i \geq i_1$*
2. *if there exists $0 \leq i_0 \leq p$ such that $x_{i_0}^* = 1$, then $x_i^* = 1$ for all $i \leq i_0$*

Proof Part 1 Consider a given equilibrium $\boldsymbol{x}^* \in \mathcal{F}_\Delta$. Assume that $x_{i_1}^* < 1$ and consider $i \geq i_1$. Since $x_{i_1}^* < 1$, this means that the equity of bank i_1 at equilibrium is positive, that is $\Delta_{i_1}^{fail} - \Delta(\boldsymbol{x}^*) > 0$. Since $\Delta_i^{fail} \geq \Delta_{i_1}^{fail}$, it thus follows that for each $i \geq i_1$, $\Delta_{i_1}^{fail} - \Delta(\boldsymbol{x}^*) \geq 0$. Part 2. Assume that $x_{i_0}^* = 1$ and consider $i \leq i_0$. As $x_{i_0}^* = 1$, this means that the equity of bank i_0 at equilibrium is equal to zero, that is $\Delta_{i_0}^{fail} - \Delta(\boldsymbol{x}^*) \leq 0$. Since $\Delta_i^{fail} \leq \Delta_{i_0}^{fail}$, it thus follows that for each $i \leq i_0$, $\Delta_{i_0}^{fail} - \Delta(\boldsymbol{x}^*) \leq 0$ $\square$

That D^* and S^*, when non empty, are composed with consecutive integers is a direct consequences of the above lemmas and note that the existence of $i^* := i(\boldsymbol{x}^*)$ depends on the equilibrium $\boldsymbol{x}^*$, i.e., $i(\boldsymbol{x}^*) \leq i(\boldsymbol{y}^*)$ if $\boldsymbol{x}^* \leq \boldsymbol{y}^*$ $\square$

Proof of Proposition 3 Given the assumption, we know that for all $1 \leq i \leq p$, $0 < x_i^* < 1$, which means that the capital ratio of each bank at date $t+1$ is equal to θ_{min}. From equation (26), $\theta_{i,t+1}(.) = \theta_{min}$ for all i is equivalent to

$$\frac{\Delta_i^{fail} - \Delta - \frac{Q^*}{\Phi}(1-\Delta)}{\alpha_i(1-x_i^*)(1-\Delta)\left(1-\frac{Q^*}{\Phi}\right)} = \theta_{min} \quad i = 1, 2, ..., p. \tag{62}$$

and this implies that:

$$q_i - x_i^* q_i = \frac{\left[\Delta_i^{fail} - \Delta - \frac{Q^*}{\Phi}(1-\Delta)\right] q_i}{\theta_{min}\alpha_i(1-\Delta)\left(1-\frac{Q^*}{\Phi}\right)} \quad i = 1, 2, ..., p. \tag{63}$$

and, summing for $i = 1$ to p, we obtain that:

$$\sum_{i=1}^{p} q_i - Q^* = \sum_{i=1}^{p} \frac{\left[\Delta_i^{fail} - \Delta - \frac{Q^*}{\Phi}(1-\Delta)\right] q_i}{\theta_{min}\alpha_i(1-\Delta)\left(1-\frac{Q^*}{\Phi}\right)} \tag{64}$$

Equation (64) remains an implicit function as Q^* appears both on the lfs and rhs. Writing (64) as $Q^* + \sum_{i=1}^{p} \frac{\left[\Delta_i^{fail} - \Delta - \frac{Q^*}{\Phi}(1-\Delta)\right] q_i}{\theta_{min}\alpha_i(1-\Delta)\left(1-\frac{Q^*}{\Phi}\right)} - \sum_{i=1}^{p} q_i = 0$, it reduces to a quadratic equation in Q^*. It thus follows that Q^* is the root of a quadratic equation that actually depends on Φ in a smooth manner, which implies that Q^* is a smooth function (i.e., regular) of Φ. Recall that when $\Phi = \infty$, we know that $Q^* = \underline{Q}^* = \sum_{i=1}^{p} \underline{x_i}^* q_i$ so that, from equation (38), we have:

$$\underline{Q}^* = \sum_{i=1}^{p} q_i - \sum_{i=1}^{p} \frac{(\Delta_i^{fail} - \Delta) q_i}{\theta_{min}\alpha_i(1-\Delta)} \tag{65}$$

Using the fact that $Q^* \geq \underline{Q}^*$ and that Q^* is a smooth function of Φ such that Q^* tends to $\underline{Q}^*$ when Φ tends to infinity, we thus can write:

$$Q^* = \underline{Q}^* \left(1 + \frac{\gamma}{\Phi}\right) + o\left(\frac{1}{\Phi}\right) \tag{66}$$

for some positive γ. Reintroducing the expression of Q^* given in equation (66) into (64), and neglecting terms of $o(\frac{1}{\Phi^2})$, it is not difficult (but cumbersome) to identify the term γ to find equation (45). Using equation (63), we then find equation (44) $\square$

8 Appendix B : tables and balance sheet split by trading and banking books

8.1 Tables

Data from Bloomberg as of 2014.

Bank	Total Capital	RWA	Total Assets	RBC
Ally Financial Inc	17294	130590	151631	0,132
American Express Company	20800	133300	159103	0,156
Bank of America Corporation	161623	1262000	2104534	0,147
BB&T Corporation	21381	143675	186834	0,149
BBVA Compass Bancshares, Inc	8254.2	64417.8	83152.4	0,128
BMO Financial Corp	31927	222931	588659	0,143
Capital One Financial Corporation	35886	237587	308167	0,151
Citigroup Inc	165454	1292605	1842181	0,128
Citizens Financial Group Inc	16781	105964	132857	0,158
Comerica Incorporated	7169	68273	69190	0,105
Discover Financial Services	12418	72744.9	83126	0,171
Fifth Third Bancorp	16895	117878	138706	0,143
HSBC North America Holdings Inc	190730	1219800	2634139	0,156
Huntington Bancshares Incorporated	7388	54479	66298	0,136
JPMorgan Chase &Co	206594	1619287	2572274	0,128
KeyCorp	11824	85100	93821	0,139
M&T Bank Corporation	11767.3	77365.6	96685.5	0,152
Morgan Stanley	74972	456008	801510	0,164
MUFG Americas Holdings Corporation	14246	96663	113623	0,147
Northern Trust Corporation	9449.2	62896.9	109946.5	0,150
Regions Financial Corporation	15070	98974	119563	0,152
Santander Holdings USA, Inc	9872.6	66751.9	118457.4	0,148
State Street Corporation	17914	107827	274119	0,166
SunTrust Banks, Inc	20338	162500	190328	0,125
The Bank of New York Mellon	21556	168028	385303	0,128
The Goldman Sachs Group, Inc	90978	570313	856240	0,160
The PNC Financial Services Group, Inc	44782	281379	345072	0,159
U.S. Bancorp	43208	317705.9	402529	0,136
Wells Fargo & Company	192900	1242500	1687155	0,155
Zions Bancorporation	7443.301	45738	57208.8	0,163

Table 7: All quantities except the RBC are in million of dollars

Fraction of insolvent banks at equilibrium as a function of Δ and Φ

Δ || $\frac{Q_{tot}}{\Phi}$	0	1%	3%	5%	6.75%	8.5%	10%	11.75%	15%
0.01	0	0	0	0	0	0	0	0	0
0.02	0	0	0	0	0	0	0	0	0
0.03	0	0	0	0	0,33	0	0	0	1
0.04	0	0	0	0.066	0,53	0.66	0.96	1	1
0.05	0	0	0.1	0.3	0.73	0.86	0.96	1	1
0.06	0.066	0.066	0.233	0.4	0.96	0.96	1	1	1
0.07	0.1	0.133	0.333	0.633	0.96	0.96	1	1	1
0.08	0.166	0.266	0.5	0.833	1	1	1	1	1
0.09	0.3	0.33	0.666	0.966	1	1	1	1	1
0.1	0.366	0.5	0.933	0.966	1	1	1	1	1
0.11	0.5	0.666	0.966	1	1	1	1	1	1
0.12	0.666	0.86	1	1	1	1	1	1	1
0.13	0.9	0.966	1	1	1	1	1	1	1
0.14	0.966	0.9667	1	1	1	1	1	1	1
0.15	1	1	1	1	1	1	1	1	1

Table 8: Fraction of insolvent banks

Best response of each bank when $\Delta = 6\%$.

Bank	$\frac{Q_{tot}}{\Phi} = 0$	$\frac{Q_{tot}}{\Phi} = 1\%$	$\frac{Q_{tot}}{\Phi} = 3\%$	$\frac{Q_{tot}}{\Phi} = 5\%$	$\frac{Q_{tot}}{\Phi} = 15\%$
Ally Financial Inc	0.16	0.23	0.54	0.84	1
American Express Company	0	0	0.26	0.55	1
Bank of America Corporation	0.62	0.73	1	1	1
BB&T Corporation	0.05	0.13	0.48	0.81	1
BBVA Compass Bancshares, Inc	0.32	0.40	0.76	1	1
BMO Financial Corp	1	1	1	1	1
Capital One Financial Corporation	0.02	0.10	0.45	0.78	1
Citigroup Inc	0.43	0.52	0.91	1	1
Citizens Financial Group Inc	0	0	0.30	0.61	1
Comerica Incorporated	0.41	0.47	0.75	1	1
Discover Financial Services	0	0	0	0.28	1
Fifth Third Bancorp	0.03	0.10	0.41	0.71	1
HSBC North America Holdings Inc	0.64	0.78	1	1	1
Huntington Bancshares Incorporated	0.16	0.24	0.57	0.87	1
JPMorgan Chase &Co	0.57	0.67	1	1	1
KeyCorp	0.03	0.09	0.39	0.66	1
M&T Bank Corporation	0	0.05	0.38	0.69	1
Morgan Stanley	0.21	0.32	0.81	1	1
MUFG Americas Holdings Corporation	0	0.04	0.36	0.65	1
Northern Trust Corporation	0.39	0.50	0.99	1	1
Regions Financial Corporation	0	0.01	0.33	0.63	1
Santander Holdings USA, Inc	0.44	0.56	1	1	1
State Street Corporation	0.81	0.98	1	1	1
SunTrust Banks, Inc	0.27	0.34	0.66	0.95	1
The Bank of New York Mellon	1	1	1	1	1
The Goldman Sachs Group, Inc	0.07	0.16	0.57	0.95	1
The PNC Financial Services Group, Inc	0	0	0.26	0.56	1
U.S. Bancorp	0.20	0.28	0.62	0.94	1
Wells Fargo & Company	0.01	0.10	0.46	0.81	1
Zions Bancorporation	0	0	0.24	0.54	1

Table 9: Liquidated proportions x_i^* for $\Delta = 6\%$

8.2 Balance sheet split by banking and trading books

From equations (49) and (50), abstracting time and bank subscripts for notational simplicity, the regulatory weights can be implied as follows

$$\alpha = \frac{\text{RWA}^{\text{Bank}}}{V^{\text{Bank}}} \tag{67}$$

$$\beta = \frac{\text{RWA}^{\text{Trad}}}{V^{\text{Trad}}} \tag{68}$$

However, in practice, the balance sheet split by trading and banking books is in general not available so that V^{Bank} and V^{Trad} but also RWA^{Bank} and RWA^{Trad} are not directly observable. To estimate the two (average) regulatory weights α and β, we first need to construct all these unobservable quantities.

For depositary institutions with an important lending activity, such as Bank of America, Citigroup, JP Morgan and Wells Fargo, the item "loans" is typically the most important one of the banking book. Regarding now the trading book, the most important items of the balance sheet are "trading assets" and "investment securities", subject to market risk and and to counterparty risk. It thus follows that (up to additional more minor items), V^{Bank} and RWA^{Bank} can be approximated respectively by the value of the loans reported in the balance sheet and the credit risk-weighted assets disclosed in the annual report. In the same vein, V^{Trad} and RWA^{Trad} can be approximated respectively using the reported value of the items trading assets and investment securities and the market risk-weighted assets disclosed in the annual report. When counterparty risk-weighted assets are disclosed, as it is related to the trading activity of the bank, we have chosen to assign those risk-weighted assets to the trading book risk-weighted assets. Under Basel 3, apart the credit and the market risk-weighted assets, banks also disclose risk-weighted assets related to operational risk. This partial risk-weighted assets is more delicate to allocate as both lending and trading activities are subject to operational risk. As a simple but reasonable rule of thumb, we make the assumption that the operational risk-weighted assets is proportional to V^{Bank} and V^{Trad}. Let

$$f = \frac{V^{\text{Bank}}}{V^{\text{Bank}} + V^{\text{Trad}}} \tag{69}$$

To obtain the risk-weighted assets of the banking book (trading book), we add to the credit risk-weighted assets (market risk-weighted assets) a fraction f $(1-f)$ of the operational risk-weighted assets. Under this assumption, the banking book risk-weighted assets and the trading book risk-weighted assets are computed as follows.

$$\text{RWA}^{\text{Bank}} = \text{RWA}^{\text{Credit}} + f\ \text{RWA}^{\text{Operat.}} \tag{70}$$

$$\text{RWA}^{\text{Trad}} = \text{RWA}^{\text{Market}} + (1-f)\ \text{RWA}^{\text{Operat.}} \tag{71}$$

where, as explained

$$V^{\text{Bank}} \approx \text{Loans} \tag{72}$$

$$V^{\text{Trad}} \approx \text{Trading assets} + \text{Investment Securities} \tag{73}$$

Minor items such as goodwill, premises and equipment, and other assets have been assigned to the banking booking.

Calibrated quantities (in billion)

BHC‖Quantities	V^{Trad}	V^{Bank}	RWA^{Trad}	RWA^{Bank}	α	β
Bank of America	565.2	1400.7	279.4	1185.6	0.494	0.846
Citigroup	596.9	1213.17	203.5	1089.1	0.341	0.898
The Goldman Sachs	473.97	324.668	335.91	234.5	0.708	0.722
JP Morgan Chase& Co	857.4	1687.9	313.4	1305.6	0.365	0.773
Morgan Stanley	430.72	349.4	204.04	251.98	0.4737	0.7211
Wells Fargo & Company	355.95	1311.61	130.24	1115.26	0.366	0.85

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