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Plant Capacity Notions: Existence Results at Firm and Industry Levels

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Abstract

This contribution innovates by investigating the question as to the existence of solutions for the key plant capacity concepts using general nonparametric technologies. Focusing on both short-run and long-run output-oriented, attainable output-oriented, and input-oriented plant capacity notions, we first investigate the existence of solutions at the firm level. Then, for this same range of six plant capacity concepts, we also explore the more difficult question as to the existence of solutions at the industry level.

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Keywords: Data Envelopment Analysis; Nonparametric Technology; Capacity utilization

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1 Introduction

Johansen (1968) is probably the first to introduce a technical or engineering approach to capacity measurement by proposing the plant capacity concept in the economic literature using single output production functions. In particular, Johansen (1968) informally defines plant capacity by the maximal amount of output that can be produced per unit of time with existing plants and equipment without restrictions on the amount of available variable inputs. On the one hand, Färe (1988) and Färe, Grosskopf, and Kokkelenberg (1989) and on the other hand Färe, Grosskopf, and Valdmanis (1989) provide an operational way to measure this plant capacity notion using a nonparametric frontier framework focusing on a single output and multiple outputs, respectively. Using a general specification of a nonparametric frontier technology, plant capacity utilisation can then be determined from observed input and output data by calculating a couple of output-oriented efficiency measures. This framework has been applied in a series of empirical applications mainly in the health care (e.g., Karagiannis (2015)) and in the fisheries industries (for instance, Felthoven (2002)).

Kerstens, Sadeghi, and Van de Woestyne (2019b) argue and empirically illustrate that the above notion of output-oriented plant capacity is unrealistic in that the amounts of variable inputs needed to reach the maximum capacity outputs may be unavailable at either the firm or the industry level. This criticism goes back to the so-called attainability issue already described in Johansen (1968). To remedy this problem, Kerstens, Sadeghi, and Van de Woestyne (2019b) propose a new attainable output-oriented plant capacity notion that bounds the available amount of variable inputs.

Alternatively, Cesaroni, Kerstens, and Van de Woestyne (2017) adopt the same nonparametric frontier framework to propose a new input-oriented measure of plant capacity utilisation based on a pair of input-oriented efficiency measures. Their empirical illustration reveals that traditional output-oriented and new input-oriented plant capacity concepts measure different things and lead to different rankings.

Cesaroni, Kerstens, and Van de Woestyne (2019) define new long-run output-oriented as well as input-oriented plant capacity concepts: these allow for changes in all input dimensions simultaneously rather than solely allowing for changes in the variable input dimensions. The plant capacity concepts focusing on changes in the variable inputs alone can then be re-interpreted as short-run plant capacity concepts.

These various short-run and long-run output-oriented and input-oriented plant capacity measures have been empirically applied to measure hospital capacity in the Hubei province in China during the recent COVID epidemic in Kerstens and Shen (2021). Though the sample is limited, the empirical evidence indicates that the long-run input-oriented plant capacity notion correlates best with the observed mortality. This may lead empirical researchers to reconsider their choice of plant capacity concept.

This contribution sets itself two main objectives. First, all of the above contributions assume the existence of results for the required efficiency measures within the nonparametric frontier framework. Apart from the result in Färe (1984) who shows that output-oriented plant capacity cannot be obtained for certain popular parametric specifications of a single output production function (e.g., the CES production function under certain parameter restrictions), no existence results exist for general nonparametric frontier technologies at the firm level. Equally so, no existence results is known to us at the level of the industry. While the seminal contributions of Färe (1988), Färe, Grosskopf, and Kokkelenberg (1989) and Färe, Grosskopf, and Valdmanis (1989) determine plant capacity on constant returns to scale technologies, Kerstens and Shen (2021) instead favour the use of variable returns to scale technologies and identify four other hospital capacity studies doing similarly (e.g., Karagiannis (2015)). Irrespective of the proper choice of returns to scale assumption in the context of plant capacity measurement, it should be noted that the input-oriented measure of plant capacity utilisation has so far only been defined for variable returns to scale technologies. It is an open question whether it can be defined relative to constant returns to scale technologies.

Second, we add some additional interpretations on how these plant capacity concepts can be used in empirical applications. Furthermore, we explicitly spell out the implications of these various plant capacity notions in terms of the availability of input and output data to empirically estimate these notions using nonparametric frontier technologies.

This contribution is structured in the following way. Section 2 prepares the floor by defining general technologies, the required nonparametric frontier technologies as well as the necessary efficiency measures. The next Section 3 defines the various short-run and long-run plant capacity notions and proves their existence at the firm level. This leads to the definition of a new long-run attainable output-oriented plant capacity concept. The following Section 4 verifies whether these same plant capacity concepts also exist at the industry level. This is first done for the short-run concepts and we indicate how the results transpose to the long-run plant capacity concepts. The next Section 5 discusses some numerical issues related to the definition of some plant capacity concepts under a constant returns to scale assumption. Conclusions wrap up the main results in Section 6.

2 Technology and Efficiency Measures: Definitions

2.1 Technology: Definitions and Axioms

We start by defining the technology and some basic notation. We start from a given N-dimensional input vector $\mathbf{x} \in \mathbb{R}^N_+$ and an M-dimensional output vector $\mathbf{y} \in \mathbb{R}^M_+$. The production possibility set or technology T is defined as $T = \{(\mathbf{x}, \mathbf{y}) \mid \mathbf{x} \text{ can produce at least } \mathbf{y}\}$.[1] Commonly, the following

[1] Throughout this contribution, $\mathbb{R}^d$ denotes the d-dimensional Euclidean space, and $\mathbb{R}^d_+$ denotes its non-negative orthant; lowercase boldface letters are used to denote vectors; all vectors are considered to be column vectors and

conditions are imposed on the input and output data defining the technology (see, e.g., Färe, Grosskopf, and Lovell (1994, p. 44-45)): (D.1) each firm utilises nonnegative amounts of each input to produce nonnegative amounts of each output; (D.2) there exists an aggregate production of positive amounts of every output, and an aggregate use of positive amounts of every input; and (D.3) each firm uses a positive amount of at least one input to produce a positive amount of at least one output.

Associated with this technology T, the input set $L(\mathbf{y}) = \{\mathbf{x} \mid (\mathbf{x}, \mathbf{y}) \in T\}$ contains all input vectors $\mathbf{x}$ that yield at least a given output vector $\mathbf{y}$. Similarly, associated with technology T one can define an output set $P(\mathbf{x}) = \{\mathbf{y} \mid (\mathbf{x}, \mathbf{y}) \in T\}$ that contains all output vectors $\mathbf{y}$ that can be generated from at most a given input vector $\mathbf{x}$.

The technology T, the input set $L(\mathbf{y})$ and the output set $P(\mathbf{x})$ are related as follows (Färe (1988, p. 5)):

$$(\mathbf{x}, \mathbf{y}) \in T \Longleftrightarrow \mathbf{x} \in L(\mathbf{y}) \Longleftrightarrow \mathbf{y} \in P(\mathbf{x}). \tag{1}$$

Though the input set, the output set as well as the technology represent the same production technology, each highlights a different aspect. The input set focuses on input substitution, the output set centers on output substitution, and the technology T aims at the transformation of inputs into outputs (Färe (1988, p. 5)).

In our contribution, technology T respects some combination of the following axioms:

(T.1) Possibility of inaction and no free lunch, i.e., $(\mathbf{0}, \mathbf{0}) \in T$ and if $(\mathbf{0}, \mathbf{y}) \in T$, then $\mathbf{y} = \mathbf{0}$.

(T.2) T is a closed subset of $\mathbb{R}_+^N \times \mathbb{R}_+^M$.

(T.3) Strong disposal of inputs and outputs, i.e., if $(\mathbf{x}, \mathbf{y}) \in T$ and $(\mathbf{x}', \mathbf{y}') \in \mathbb{R}_+^N \times \mathbb{R}_+^M$, then $(\mathbf{x}', -\mathbf{y}') \geq (\mathbf{x}, -\mathbf{y}) \Rightarrow (\mathbf{x}', \mathbf{y}') \in T$.

(T.4) $(\mathbf{x}, \mathbf{y}) \in T \Rightarrow \delta(\mathbf{x}, \mathbf{y}) \in T$ for $\delta \in \Gamma$, where:

- (i) $\Gamma \equiv \text{CRS} = \{\delta \mid \delta \geq 0\}$;
- (ii) $\Gamma \equiv \text{VRS} = \{\delta \mid \delta = 1\}$.

(T.5) T is convex.

These traditional axioms on technology merit the following remarks (see Hackman (2008)). Production can be halted (inaction) and without inputs one cannot generate any outputs (no free lunch). The production possibility set is closed. Inputs can be wasted, and outputs can be destroyed at no opportunity costs (strong or free disposability of inputs and outputs). We consider two returns

vectors $\mathbf{0}$ denotes vector of zeroes; and for vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^d$, the inequality $\mathbf{a} \geq \mathbf{b}$ ($\mathbf{a} > \mathbf{b}$) means that $a_i \geq b_i$ ($a_i > b_i$), for all $i = 1, \ldots, d$.

to scale assumptions: either constant returns to scale (CRS), or variable returns to scale (VRS). Finally, technology is convex.

Observe that these axioms are not always maintained in this contribution.[2] Specifically, central axioms distinguishing the technologies in the empirical analysis are: (i) CRS versus VRS, and (ii) convexity versus nonconvexity.

In economics it is customary to distinguish in the short run between fixed and variable inputs depending on whether inputs are exogenous to managerial control or are fully controlled by management. This leads to a partitioning of the input vector $\mathbf{x}$ into a fixed ($\mathbf{x}^f$) and variable part ($\mathbf{x}^v$). One can denote $\mathbf{x} = (\mathbf{x}^f, \mathbf{x}^v)$ with $\mathbf{x}^f \in \mathbb{R}_+^{N_f}$ and $\mathbf{x}^v \in \mathbb{R}_+^{N_v}$ such that $N = N_f + N_v$. To simplify, it is assumed that all producers share common subvectors of fixed and variable input dimensions.

Partitioning the input vector requires sharpening the conditions on inputs and outputs. In particular, Färe, Grosskopf, and Kokkelenberg (1989, p. 659–660) state: (D.4) each fixed input is used by some firm, and each firm uses some fixed input. We also need: (D.5) each variable input is used by some firm, and each firm uses some variable input.

Based on Färe, Grosskopf, and Valdmanis (1989), we can define a short-run technology $T^f = \{(\mathbf{x}^f, \mathbf{y}) \in \mathbb{R}_+^{N_f} \times \mathbb{R}_+^M \mid$ there exist some $\mathbf{x}^v$ such that $(\mathbf{x}^f, \mathbf{x}^v)$ can produce at least $\mathbf{y}\}$ as well as the corresponding output set $P^f(\mathbf{x}^f) = \{\mathbf{y} \mid (\mathbf{x}^f, \mathbf{y}) \in T^f\}$.

2.2 Nonparametric Frontier Technologies

Consider K observations $(k = 1, \dots, K)$ with each a vector of inputs and outputs $(\mathbf{x}_k, \mathbf{y}_k) \in \mathbb{R}_+^N \times \mathbb{R}_+^M$. The corresponding convex and nonconvex nonparametric frontier technologies under the CRS and VRS assumptions, as well as the input and output sets, can be mathematically represented as follows:

$$T_{\Lambda,\Gamma} = \left\{ (\mathbf{x}, \mathbf{y}) \mid \mathbf{x} \geq \sum_{k=1}^{K} \delta z_k \mathbf{x}_k, \mathbf{y} \leq \sum_{k=1}^{K} \delta z_k \mathbf{y}_k, \mathbf{z} = (z_1, \dots, z_K) \in \Lambda, \delta \in \Gamma \right\}, \tag{2}$$

$$L_{\Lambda,\Gamma}(\mathbf{y}_p) = \left\{ \mathbf{x} \mid \mathbf{x} \geq \sum_{k=1}^{K} \delta z_k \mathbf{x}_k, \mathbf{y}_p \leq \sum_{k=1}^{K} \delta z_k \mathbf{y}_k, \mathbf{z} \in \Lambda, \delta \in \Gamma \right\}, \tag{3}$$

$$P_{\Lambda,\Gamma}(\mathbf{x}_p) = \left\{ \mathbf{y} \mid \mathbf{x}_p \geq \sum_{k=1}^{K} \delta z_k \mathbf{x}_k, \mathbf{y} \leq \sum_{k=1}^{K} \delta z_k \mathbf{y}_k, \mathbf{z} \in \Lambda, \delta \in \Gamma \right\}, \tag{4}$$

where $(\mathbf{x}_p, \mathbf{y}_p)$ is the unit under evaluation; Λ is either C or NC as follows:

[2] Note that the convex VRS technology does not satisfy inaction.

(i) $\Lambda \equiv \mathrm{C} = \left\{ \mathbf{z} \mid \sum_{k=1}^{K} z_k = 1 \text{ and } \forall k \in \{1, \ldots, K\} : z_k \geq 0 \right\};$

(ii) $\Lambda \equiv \mathrm{NC} = \left\{ \mathbf{z} \mid \sum_{k=1}^{K} z_k = 1 \text{ and } \forall k \in \{1, \ldots, K\} : z_k \in \{0, 1\} \right\},$

and Γ is either CRS or VRS as follows:

(i) $\Gamma \equiv \mathrm{CRS} = \{\delta \mid \delta \geq 0\};$

(ii) $\Gamma \equiv \mathrm{VRS} = \{\delta \mid \delta = 1\}.$

The short-run technology $T^f_{\Lambda,\Gamma}$ can be represented algebraically as follows:

$$T^f_{\Lambda,\Gamma} = \left\{ (\mathbf{x}^f, \mathbf{y}) \mid \mathbf{x}^f \geq \sum_{k=1}^{K} \delta z_k \mathbf{x}^f_k, \mathbf{x}^v \geq \sum_{k=1}^{K} \delta z_k \mathbf{x}^v_k, \mathbf{y} \leq \sum_{k=1}^{K} \delta z_k \mathbf{y}_k, \mathbf{z} \in \Lambda, \delta \in \Gamma \right\}. \tag{5}$$

The short-run output set $P^f_{\Lambda,\Gamma}(\mathbf{x}^f_p)$ is represented algebraically by:

$$P^f_{\Lambda,\Gamma}(\mathbf{x}^f_p) = \left\{ \mathbf{y} \mid \mathbf{x}^f_p \geq \sum_{k=1}^{K} \delta z_k \mathbf{x}^f_k, \mathbf{x}^v \geq \sum_{k=1}^{K} \delta z_k \mathbf{x}^v_k, \mathbf{y} \leq \sum_{k=1}^{K} \delta z_k \mathbf{y}_k, \mathbf{z} \in \Lambda, \delta \in \Gamma \right\}. \tag{6}$$

Proposition 2.1. *The variable input constraints are redundant at the firm level and can be removed from the short-run technology $T^f_{\Lambda,\Gamma}$ and the short-run output set $P^f_{\Lambda,\Gamma}(\boldsymbol{x}^f_p)$ at the firm level.*

The proof of Proposition 2.1 as well as the other statements are available in Appendix A. Based on Proposition 2.1, we can eliminate constraint $\mathbf{x}^v \geq \sum_{k=1}^{K} \delta z_k \mathbf{x}^v_k$ from (5) and (6).

Remark 2.1. *In the literature, one can find three variations on the definition of the short-run technology $T^f_{\Lambda,\Gamma}$ that are compatible with our formulation.*

- *The seminal works of Färe (1988), Färe, Grosskopf, and Kokkelenberg (1989) and Färe, Grosskopf, and Valdmanis (1989) all drop the variable input constraints from their definition of the short-run technology* (5) *and* (6)*. This can only be meaningfully interpreted if the authors implicitly have the above variable input constraints in mind whereby the amount of variable inputs are decision variables ($\boldsymbol{x}^v$). Only then, these variable input constraints are redundant.*

- *In Färe, Grosskopf, and Lovell (1994, p. 262) a related argument contains a minor typo: in our notation, it is argued that $\sum_{k=1}^{K} \delta z_k \boldsymbol{x}^v_k = \boldsymbol{\lambda} \boldsymbol{x}^v_p$ with $\boldsymbol{\lambda} \in \mathbb{R}^N_+$ and variable inputs as parameters ($\boldsymbol{x}^v_p$). But, only $\sum_{k=1}^{K} \delta z_k \boldsymbol{x}^v_k \leq \boldsymbol{\lambda} \boldsymbol{x}^v_p$ can make these variable input constraints redundant.*

- *In Cesaroni, Kerstens, and Van de Woestyne (2019, p. 388) and Kerstens, Sadeghi, and Van de Woestyne (2019a, p. 701) the short-run technology $T^f_{\Lambda,\Gamma}$ is considered as a projection of the general technology $T_{\Lambda,\Gamma}$ into the subspace of fixed inputs and outputs, i.e., technology $T^f_{\Lambda,\Gamma}$ is in fact obtained by a projection of technology $T_{\Lambda,\Gamma} \in \mathbb{R}^{N+M}_+$ into the subspace $\mathbb{R}^{N_f+M}_+$ (i.e., by setting all variable inputs equal to zero). By analogy, the same applies to the output set $P^f_{\Lambda,\Gamma}(\boldsymbol{x}^f_p)$. Note that by fixing all variable inputs to any identical numerical value one again makes the variable input constraints redundant.*

Note that the input set $L_{\Lambda,\Gamma}(\mathbf{y}_p)$ and the output set $P_{\Lambda,\Gamma}(\mathbf{x}_p)$ are nonempty and closed sets. Also, the output set $P_{\Lambda,\Gamma}(\mathbf{x}_p)$ is a bounded set. This guarantees the existence of input- and output-oriented efficiency measures (see Section 2.3). In the following Proposition 2.2 we prove that the short-run output set $P^f_{\Lambda,\Gamma}(\mathbf{x}^f_p)$ is a nonempty and compact set.

Proposition 2.2. *The short-run output set $P^f_{\Lambda,\Gamma}(\boldsymbol{x}^f_p)$ is a nonempty and compact set.*

Thus, the short-run output set $P^f_{\Lambda,\Gamma}(\mathbf{x}^f_p)$ is nonempty and compact under the convex and nonconvex assumptions as well as in the CRS and VRS cases. Therefore, Proposition 2.2 guarantees the existence of the short-run output-oriented efficiency measure (see Section 2.3).

Generalizing Cesaroni, Kerstens, and Van de Woestyne (2017, p. 727), one can define the following atypical definition: $L_{\Lambda,\Gamma}(\mathbf{0}) = \{\mathbf{x} \mid (\mathbf{x}, \mathbf{0}) \in T_{\Lambda,\Gamma}\}$ is the input set compatible with a zero output level. This input set indicates the input levels where non-zero production is initiated. The input set $L_{\Lambda,\Gamma}(\mathbf{0})$ can be obtained by (3) when we replace the output constraint $\mathbf{y}_p \leq \sum_{k=1}^{K} \delta z_k \mathbf{y}_k$ with $\mathbf{0} \leq \sum_{k=1}^{K} \delta z_k \mathbf{y}_k$.

Proposition 2.3. *The output constraints are redundant at the firm level and can be removed from the short-run input set $L_{\Lambda,\Gamma}(\boldsymbol{0})$ at the firm level.*

We introduce $L_{\Lambda,\Gamma}(\mathbf{y}_{min}) = \{\mathbf{x} \mid (\mathbf{x}, \mathbf{y}_{min}) \in T_{\Lambda,\Gamma}\}$, whereby $\mathbf{y}_{min} = \min_{k=1,\dots,K} \mathbf{y}_k$. Therefore, the minimum output is determined component-wise for every output $\mathbf{y}$ over all units K under both the C and NC cases and for the CRS and VRS axioms. Moreover, let $L_{\Lambda,\Gamma}(\mathbf{y}^\epsilon) = \{\mathbf{x} \mid (\mathbf{x}, \mathbf{y}^\epsilon) \in T_{\Lambda,\Gamma}\}$ where $\mathbf{y}^\epsilon \in \mathbb{R}^M_+$ is a vector with arbitrary small components and $\mathbf{y}^\epsilon \leq \mathbf{y}_{min}$: this inequality is compatible with the assumption of strong output disposal. Note that $L_{\Lambda,\Gamma}(\mathbf{y}^\epsilon) = \{\mathbf{x} \mid (\mathbf{x}, \mathbf{y}^\epsilon) \in T_{\Lambda,\Gamma}\}$ is the input set compatible with a $\mathbf{y}^\epsilon$ output level. This input set denotes the input levels where production is started up. Note that $L_{\Lambda,\Gamma}(\mathbf{0})$, $L_{\Lambda,\Gamma}(\mathbf{y}_{min})$ and $L_{\Lambda,\Gamma}(\mathbf{y}^\epsilon)$ are nonempty and closed sets.

Proposition 2.4. *(i) In the VRS case, we have $L_{\Lambda,VRS}(\boldsymbol{0}) = L_{\Lambda,VRS}(\boldsymbol{y}^\epsilon) = L_{\Lambda,VRS}(\boldsymbol{y}_{min}) \subset \mathbb{R}^N_+$.*

(ii) In the CRS case, we have $L_{\Lambda,CRS}(\boldsymbol{y}_{min}) \subseteq L_{\Lambda,CRS}(\boldsymbol{y}^\epsilon) \subset L_{\Lambda,CRS}(\boldsymbol{0}) = \mathbb{R}^N_+$.

Under the VRS assumption, for each output level $\mathbf{y} \leq \mathbf{y}_{min}$ we have the same input set $L_{\Lambda,VRS}(\mathbf{y})$. While under the CRS assumption, a higher output level leads to a smaller input set. Moreover, under the VRS case we have $L_{\Lambda,VRS}(\mathbf{0}) \subset \mathbb{R}^N_+$ while under the CRS case we have $L_{\Lambda,CRS}(\mathbf{0}) = \mathbb{R}^N_+$. We show in an illustrative example below how the value of $\mathbf{y}^\epsilon$ determines the quality of the solutions for the CRS case.

Extending Cesaroni, Kerstens, and Van de Woestyne (2019), we now define the particular output set $P_{\Lambda,\Gamma} = \{\mathbf{y} \mid \exists\, \mathbf{x} : (\mathbf{x},\mathbf{y}) \in T_{\Lambda,\Gamma}\}$ including all possible outputs irrespective of the needed inputs. The long-run output set $P_{\Lambda,\Gamma}$ is represented algebraically by:

$$P_{\Lambda,\Gamma} = \left\{ \mathbf{y} \mid \mathbf{x} \geq \sum_{k=1}^{K} \delta z_k \mathbf{x}_k, \mathbf{y} \leq \sum_{k=1}^{K} \delta z_k \mathbf{y}_k, \mathbf{z} \in \Lambda, \delta \in \Gamma \right\}. \tag{7}$$

Note that $P_{\Lambda,\Gamma}$ is a non-empty and closed set under both VRS and CRS cases.

Let $P^{\mathbf{x}_{max}}_{\Lambda,\Gamma} = \{\mathbf{y} \mid \exists \mathbf{x} : \mathbf{x} \leq \mathbf{x}_{max}; (\mathbf{x},\mathbf{y}) \in T_{\Lambda,\Gamma}\}$, whereby $\mathbf{x}_{max} = \max\limits_{k=1,\ldots,K} \mathbf{x}_k$. Hence, the maximum input is taken on each component for every input $\mathbf{x}$ over all observed units K under both the C and NC cases and the CRS and VRS assumptions.

Moreover, let $P^{\mathbf{x}^\epsilon}_{\Lambda,VRS} = \{y \mid \exists \mathbf{x} : \mathbf{x} \leq \mathbf{x}^\epsilon; (\mathbf{x},\mathbf{y}) \in T_{\Lambda,\Gamma}\}$ where $\mathbf{x}^\epsilon \in \mathbb{R}^M_+$ is a vector with an arbitrary components such that $\mathbf{x}^\epsilon \geq \mathbf{x}_{max}$. Note that the inequality $\mathbf{x}^\epsilon \geq \mathbf{x}_{max}$ is justified by the assumption of strong disposal of the inputs.

Then, we have the following Proposition 2.5:

Proposition 2.5. *(i) In the VRS case, we have $P_{\Lambda,VRS} = P^{\boldsymbol{x}_{max}}_{\Lambda,VRS} = P^{\boldsymbol{x}^\epsilon}_{\Lambda,VRS} \subset \mathbb{R}^M_+$.*

(ii) In the CRS case, we have $P^{\boldsymbol{x}_{max}}_{\Lambda,CRS} \subseteq P^{\boldsymbol{x}^\epsilon}_{\Lambda,CRS} \subset P_{\Lambda,CRS} = \mathbb{R}^M_+$.

Under the VRS assumption, for each upper input level $\mathbf{x} \geq \mathbf{x}_{max}$ we have the same long-run output set $P_{\Lambda,VRS}$. While under the CRS assumption, a higher upper input level leads to a larger long-run output set.

Based on Proposition 2.5(i), $P_{\Lambda,\Gamma}$ when $\Lambda = \{C, NC\}$ and $\Gamma = VRS$ can be equivalently defined by $P^{\mathbf{x}^\epsilon}_{\Lambda,VRS}$, whereby $\mathbf{x}^\epsilon \geq \mathbf{x}_{max}$. Hence, the maximum input is selected on each component for every input $\mathbf{x}$ over all observed units K under both the C and NC cases and the VRS assumption. Moreover, $P_{\Lambda,\Gamma}$ is a bounded set under the VRS case, but not under the CRS case. In fact, under the CRS case we have $P_{\Lambda,CRS} = \mathbb{R}^M_+$, while $P^{\mathbf{x}_{max}}_{\Lambda,CRS} \subseteq P^{\mathbf{x}^\epsilon}_{\Lambda,CRS} \subset \mathbb{R}^M_+$. Note that $P^{\mathbf{x}^\epsilon}_{\Lambda,\Gamma}$ and $P^{\mathbf{x}_{min}}_{\Lambda,\Gamma}$ are nonempty, closed and bounded sets.

2.3 Efficiency Measures

The radial output efficiency measure characterizes the output set $P_{\Lambda,\Gamma}(\mathbf{x})$ completely and can be defined as follows:

$$DF_o(\mathbf{x}_p, \mathbf{y}_p \mid \Lambda, \Gamma) = \max\{\varphi \mid \varphi \geq 0, \varphi \mathbf{y}_p \in P_{\Lambda,\Gamma}(\mathbf{x}_p)\}. \quad (8)$$

It is larger than or equal to unity ($DF_o(\mathbf{x}_p, \mathbf{y}_p \mid \Lambda, \Gamma) \geq 1$), with efficient production on the boundary (isoquant) of the output set $P_{\Lambda,\Gamma}(\mathbf{x}_p)$ represented by unity, and it happens to have a revenue interpretation (e.g., Färe, Grosskopf, and Lovell (1994) or Hackman (2008)).

Next, we define the efficiency measure $DF_o(\mathbf{y}_p \mid P_{\Lambda,\Gamma})$ that does not depend on a particular input vector $\mathbf{x}_p$:

$$DF_o(\mathbf{y}_p \mid P_{\Lambda,\Gamma}) = \max\{\varphi \mid \varphi \geq 0, \varphi \mathbf{y}_p \in P_{\Lambda,\Gamma}\}. \quad (9)$$

Contrary to the radial output efficiency measure (8), this efficiency measure $DF_o(\mathbf{y}_p \mid P_{\Lambda,\Gamma})$ is allowed to choose the inputs needed for maximizing φ.

Remark 2.2. *Since we have: $P_{\Lambda,VRS} \subset \mathbb{R}^M_+$ while $P_{\Lambda,CRS} = \mathbb{R}^M_+$. As a result, we have $1 \leq DF_o(\boldsymbol{y}_p \mid P_{\Lambda,VRS}) < \infty$ and $DF_o(\boldsymbol{y}_p \mid P_{\Lambda,CRS}) = \infty$. Therefore, $DF_o(\boldsymbol{y}_p \mid P_{\Lambda,\Gamma})$ exists under the VRS assumption, but it does not exist under the CRS case.*

The next proposition illustrates the relation among the values of $DF_o(\mathbf{y}_p \mid P_{\Lambda,\Gamma})$, $DF_o(\mathbf{y}_p \mid P^{\mathbf{x}_{max}}_{\Lambda,\Gamma})$ and $DF_o(\mathbf{y}_p \mid P^{\mathbf{x}^\epsilon}_{\Lambda,\Gamma})$ when $\Gamma = VRS$ and $\Gamma = CRS$, respectively.

Proposition 2.6. *We have:*

(i) $DF_o(\boldsymbol{y}_p \mid P_{\Lambda,VRS}) = DF_o(\boldsymbol{y}_p \mid P^{\boldsymbol{x}_{max}}_{\Lambda,VRS}) = DF_o(\boldsymbol{y}_p \mid P^{x^\epsilon}_{\Lambda,VRS})$;

(ii) $DF_o(\boldsymbol{y}_p \mid P^{\boldsymbol{x}_{max}}_{\Lambda,CRS}) \leq DF_o(\boldsymbol{y}_p \mid P^{x^\epsilon}_{\Lambda,CRS}) < DF_o(\boldsymbol{y}_p \mid P_{\Lambda,CRS})$.

Under the VRS assumption, for each input level $\mathbf{x} \geq \mathbf{x}_{max}$ we have exactly the same long-run output efficiency measure $DF_o(\mathbf{y}_p \mid P_{\Lambda,VRS})$. While under the CRS assumption, higher input bounds lead to a bigger long-run output-oriented efficiency measure, with an ∞ efficiency measure for $P_{\Lambda,CRS}$. Therefore, the long-run output-oriented efficiency measure $DF_o(\mathbf{y}_p \mid P_{\Lambda,VRS})$ can be equivalently formulated as $DF_o(\mathbf{y}_p \mid P^{\mathbf{x}^\epsilon}_{\Lambda,VRS})$. We define the long-run output-oriented efficiency measure under both VRS and CRS cases as follows:

$$DF_o(\mathbf{y}_p \mid P^{\mathbf{x}^\epsilon}_{\Lambda,\Gamma}) = \max\{\varphi \mid \varphi \geq 0, \varphi \mathbf{y}_p \in P^{\mathbf{x}^\epsilon}_{\Lambda,\Gamma}\}. \quad (10)$$

Based on Proposition 2.6 and Remark 2.2, $DF_o(\mathbf{y}_p \mid P^{\mathbf{x}^\epsilon}_{\Lambda,\Gamma}) < \infty$ under both CRS and VRS cases.

Denoting the radial output efficiency measure of the short-run output set $P^f_{\Lambda,\Gamma}(\mathbf{x}^f_p)$ by $DF^f_o(\mathbf{x}^f_p, \mathbf{y}_p \mid$

$\Lambda, \Gamma)$, this short-run output-oriented efficiency measure is defined in the following way:

$$DF_o^f(\mathbf{x}_p{}^f, \mathbf{y}_p \mid \Lambda, \Gamma) = \max\{\varphi \mid \varphi \geq 0, \varphi\mathbf{y}_p \in P^f_{\Lambda,\Gamma}(\mathbf{x}_p^f)\}. \tag{11}$$

Remark 2.3. *Note that based on Proposition 2.2, since $P^f_{\Lambda,\Gamma}(\boldsymbol{x}_p^f)$ is a compact set, then this short-run output-oriented efficiency measure $DF_o^f(\boldsymbol{x}_p^f, \boldsymbol{y}_p \mid \Lambda, \Gamma)$ always exists.*

The radial input efficiency measure completely characterizes the input set $L_{\Lambda,\Gamma}(\mathbf{y}_p)$ and can be defined as follows:

$$DF_i(\mathbf{x}_p, \mathbf{y}_p \mid \Lambda, \Gamma) = \min\{\theta \mid \theta \geq 0, \theta\mathbf{x}_p \in L_{\Lambda,\Gamma}(\mathbf{y}_p)\}. \tag{12}$$

It is smaller than or equal to unity ($DF_i(\mathbf{x}_p, \mathbf{y}_p \mid \Lambda, \Gamma) \leq 1$), with efficient production on the boundary (isoquant) of $L_{\Lambda,\Gamma}(\mathbf{y}_p)$ represented by unity, and it has a cost interpretation (see, e.g., Färe, Grosskopf, and Lovell (1994) or Hackman (2008)).

Only reducing the variable inputs, a sub-vector input efficiency measure $DF_i^{SR}(\mathbf{x}_p^f, \mathbf{x}_p^v, \mathbf{y}_p \mid \Lambda, \Gamma)$ is defined as follows:

$$DF_i^{SR}(\mathbf{x}_p^f, \mathbf{x}_p^v, \mathbf{y}_p \mid \Lambda, \Gamma) = \min\{\theta \mid \theta \geq 0, (\mathbf{x}_p^f, \theta\mathbf{x}_p^v) \in L_{\Lambda,\Gamma}(\mathbf{y}_p)\}. \tag{13}$$

Remark 2.4. *The corresponding model of the input-oriented efficiency measures* (13) *is feasible and we have* $0 < DF_i^{SR}(\boldsymbol{x}^f, \boldsymbol{x}^v, \boldsymbol{y} \mid \Lambda, \Gamma) \leq 1$.

Reducing all inputs, an input-oriented efficiency measure $DF_i(\mathbf{x}_p, \mathbf{0} \mid \Lambda, \Gamma)$ relative to the input set with zero output level is given by:

$$DF_i(\mathbf{x}_p, \mathbf{0} \mid \Lambda, \Gamma) = \min\{\theta \mid \theta \geq 0, \theta\mathbf{x}_p \in L_{\Lambda,\Gamma}(\mathbf{0})\}. \tag{14}$$

Reducing variable inputs only, a sub-vector input efficiency measure $DF_i^{SR}(\mathbf{x}_p^f, \mathbf{x}_p^v, \mathbf{0} \mid \Lambda, \Gamma)$ evaluated relative to the input set with a zero output level is defined as follows:

$$DF_i^{SR}(\mathbf{x}_p^f, \mathbf{x}_p^v, \mathbf{0} \mid \Lambda, \Gamma) = \min\{\theta \mid \theta \geq 0, (\mathbf{x}_p^f, \theta\mathbf{x}_p^v) \in L_{\Lambda,\Gamma}(\mathbf{0})\}. \tag{15}$$

This sub-vector efficiency measure is defined with respect to the input set with zero output level where production is initiated. The following proposition shows that $DF_i(\mathbf{x}_p, \mathbf{0} \mid \Lambda, VRS)$ and $DF_i^{SR}(\mathbf{x}_p^f, \mathbf{x}_p^v, \mathbf{0} \mid \Lambda, VRS)$ are positive and smaller than unity under VRS and they are zero in the CRS case.

Proposition 2.7. *We have:*

(i) $0 < DF_i(\boldsymbol{x}_p, \boldsymbol{0} \mid \Lambda, VRS) \leq 1$ *and* $0 < DF_i^{SR}(\boldsymbol{x}_p^f, \boldsymbol{x}_p^v, \boldsymbol{0} \mid \Lambda, VRS) \leq 1$

(ii) $DF_i(\boldsymbol{x}_p, \boldsymbol{0} \mid \Lambda, CRS) = DF_i^{SR}(\boldsymbol{x}_p^f, \boldsymbol{x}_p^v, \boldsymbol{0} \mid \Lambda, CRS) = 0.$

The long-run and short-run input-oriented efficiency measures (14) and (15) are feasible under CRS and VRS, and we have $0 < DF_i(\mathbf{x}_p, \mathbf{0} \mid \Lambda, VRS) \leq 1$ and $0 < DF_i^{SR}(\mathbf{x}_p^f, \mathbf{x}_p^v, \mathbf{0} \mid \Lambda, VRS) \leq 1$. But, they are equal to zero under CRS.

The next proposition illustrates the relation among the values of $DF_i(\mathbf{x}_p, \mathbf{y} \mid \Lambda, \Gamma)$ when $\mathbf{y} = \mathbf{0}, \mathbf{y}^\epsilon$ and $\mathbf{y}_{min}$ respectively.

Proposition 2.8. *We have:*

(i) $DF_i(\boldsymbol{x}_p, \boldsymbol{0} \mid \Lambda, VRS) = DF_i(\boldsymbol{x}_p, \boldsymbol{y}^\epsilon \mid \Lambda, VRS) = DF_i(\boldsymbol{x}_p, \boldsymbol{y}_{min} \mid \Lambda, VRS)$;

(ii) $DF_i(\boldsymbol{x}_p, \boldsymbol{0} \mid \Lambda, CRS) < DF_i(\boldsymbol{x}_p, \boldsymbol{y}^\epsilon \mid \Lambda, CRS) < DF_i(\boldsymbol{x}_p, \boldsymbol{y}_{min} \mid \Lambda, CRS)$.

Under the VRS assumption, for each output level $\mathbf{y} \leq \mathbf{y}_{min}$ we have exactly the same input efficiency measure $DF_i(\mathbf{x}_p, \mathbf{y}_{min} \mid \Lambda, VRS)$. While under the CRS assumption, higher output levels lead to a bigger long-run input efficiency measure. Therefore, the long-run input efficiency measure $DF_i(\mathbf{x}_p, \mathbf{0} \mid \Lambda, VRS)$ can be equivalently formulated as either $DF_i(\mathbf{x}_p, \mathbf{y}_{min} \mid \Lambda, VRS)$ or $DF_i(\mathbf{x}_p, \mathbf{y}^\epsilon \mid \Lambda, VRS)$. We define the long-run input efficiency measure under both VRS and CRS cases as follows:

$$DF_i(\mathbf{x}_p, \mathbf{y}^\epsilon \mid \Lambda, \Gamma) = \min\{\theta \mid \theta \geq 0, \theta\mathbf{x}_p \in L_{\Lambda,\Gamma}(\mathbf{y}^\epsilon)\}. \tag{16}$$

The following proposition illustrates the relation among the values of $DF_i^{SR}(\mathbf{x}_p^f, \mathbf{x}_p^v, \mathbf{y} \mid \Lambda, \Gamma)$ when $\mathbf{y} = \mathbf{0}, \mathbf{y}^\epsilon$ and $\mathbf{y}_{min}$ respectively.

Proposition 2.9. *We have:*

(i) $DF_i^{SR}(\boldsymbol{x}_p^f, \boldsymbol{x}_p^v, \boldsymbol{0} \mid \Lambda, VRS) = DF_i^{SR}(\boldsymbol{x}_p^f, \boldsymbol{x}_p^v, \boldsymbol{y}^\epsilon \mid \Lambda, VRS) = DF_i^{SR}(\boldsymbol{x}_p^f, \boldsymbol{x}_p^v, \boldsymbol{y}_{min} \mid \Lambda, VRS)$;

(ii) $DF_i^{SR}(\boldsymbol{x}_p^f, \boldsymbol{x}_p^v, \boldsymbol{0} \mid \Lambda, CRS) < DF_i^{SR}(\boldsymbol{x}_p^f, \boldsymbol{x}_p^v, \boldsymbol{y}^\epsilon \mid \Lambda, CRS) \leq DF_i^{SR}(\boldsymbol{x}_p^f, \boldsymbol{x}_p^v, \boldsymbol{y}_{min} \mid \Lambda, CRS)$.

Under the VRS assumption, for each output level $\mathbf{y} \leq \mathbf{y}_{min}$ we have exactly the same short-run input efficiency measure $DF_i^{SR}(\mathbf{x}^f, \mathbf{x}^v, \mathbf{y}_{min} \mid \Lambda, VRS)$. While under the CRS assumption, higher output levels lead to a bigger input efficiency measure. Therefore, this sub-vector input efficiency measure $DF_i^{SR}(\mathbf{x}^f, \mathbf{x}^v, \mathbf{0} \mid \Lambda, VRS)$ is formulated equivalently as either $DF_i^{SR}(\mathbf{x}^f, \mathbf{x}^v, \mathbf{y}_{min} \mid \Lambda, VRS)$ or $DF_i^{SR}(\mathbf{x}^f, \mathbf{x}^v, \mathbf{y}^\epsilon \mid \Lambda, VRS)$. We define the short-run input efficiency measure under both VRS and CRS cases as follows:

$$DF_i^{SR}(\mathbf{x}_p^f, \mathbf{x}_p^v, \mathbf{y}^\epsilon \mid \Lambda, \Gamma) = \min\{\theta \mid \theta \geq 0, (\mathbf{x}_p^f, \theta\mathbf{x}_p^v) \in L_{\Lambda,\Gamma}(\mathbf{y}^\epsilon)\}. \tag{17}$$

3 Plant Capacity Concepts at the Firm Level

3.1 Short-run Plant Capacity Concepts

Recalling the informal plant capacity definition by Johansen (1968, p. 362) as "the maximum amount that can be produced per unit of time with existing plant and equipment, provided that the availability of variable factors of production is not restricted", this output-oriented plant capacity notion is made operational by Färe (1988), Färe, Grosskopf, and Kokkelenberg (1989) and Färe, Grosskopf, and Valdmanis (1989) using a couple of output-oriented efficiency measures. We now recall the formal definition of this output-oriented plant capacity utilization.

Definition 3.1. The short-run output-oriented plant capacity utilization PCU_o^{SR} is defined as follows:

$$PCU_o^{SR}(\mathbf{x}_p, \mathbf{x}_p^f, \mathbf{y}_p \mid \Lambda, \Gamma) = \frac{DF_o(\mathbf{x}_p, \mathbf{y}_p \mid \Lambda, \Gamma)}{DF_o^f(\mathbf{x}_p^f, \mathbf{y}_p \mid \Lambda, \Gamma)},$$

where $DF_o(\mathbf{x}_p, \mathbf{y}_p \mid \Lambda, \Gamma)$ and $DF_o^f(\mathbf{x}_p^f, \mathbf{y}_p \mid \Lambda, \Gamma)$ are output efficiency measures including, respectively excluding, the variable inputs as defined before in (8) and (11).

Since $1 \leq DF_o(\mathbf{x}_p, \mathbf{y}_p \mid \Lambda, \Gamma) \leq DF_o^f(\mathbf{x}_p^f, \mathbf{y}_p \mid \Lambda, \Gamma)$, notice that $0 < PCU_o^{SR}(\mathbf{x}_p, \mathbf{x}_p^f, \mathbf{y}_p \mid \Lambda, \Gamma) \leq 1$. Thus, short-run output-oriented plant capacity utilization has an upper limit of unity. As a result, we have the following remark.

Remark 3.1. *Note that Färe (1988, p. 70) shows that if we have an upper bound on the fixed inputs, then the short-run output-oriented plant capacity $PCU_o^{SR}(\boldsymbol{x}_p, \boldsymbol{x}_p^f, \boldsymbol{y}_p \mid \Lambda, CRS)$ exists at the firm level under CRS and a single output. Therefore, constraints $\boldsymbol{x}^f \geq \sum_{k=1}^{K} \delta z_k \boldsymbol{x}_k^f$ of fixed inputs in* (5) *guarantee that the short-run output-oriented efficiency measure $DF_o^f(\boldsymbol{x}_p^f, \boldsymbol{y}_p \mid \Lambda, CRS)$ exists and therefore $PCU_o^{SR}(\boldsymbol{x}_p, \boldsymbol{x}_p^f, \boldsymbol{y}_p \mid \Lambda, CRS)$ also exists. If we do not have any fixed inputs, i.e., all inputs are variable (in case that data property (D.4) is not respected by the data), then there is no guarantee that $PCU_o^{SR}(\boldsymbol{x}_p, \boldsymbol{x}_p^f, \boldsymbol{y}_p \mid \Lambda, CRS)$ exists under CRS (see also the long-run output-oriented plant capacity notion that is addressed in Section 3.2). As a result, the short-run output-oriented plant capacity $PCU_o^{SR}(\boldsymbol{x}_p, \boldsymbol{x}_p^f, \boldsymbol{y}_p \mid \Lambda, \Gamma)$ exists at the firm level under both the VRS and CRS cases as well as under both the convex and nonconvex assumptions.*

Depending on whether one disregards inefficiency or accommodates for the eventual existence of inefficiency, Färe, Grosskopf, and Kokkelenberg (1989) distinguish between a so-called biased and an unbiased plant capacity measure $DF_o^f(\mathbf{x}_p^f, \mathbf{y}_p \mid \Lambda, \Gamma)$ and $PCU_o^{SR}(\mathbf{x}_p, \mathbf{x}_p^f, \mathbf{y}_p \mid \Lambda, \Gamma)$, respectively. The latter unbiased plant capacity measures $PCU_o^{SR}(\mathbf{x}_p, \mathbf{x}_p^f, \mathbf{y}_p \mid \Lambda, \Gamma)$ as a ratio of efficiency measures yields a cleaned notion of output-oriented plant capacity by removing any existing inefficiency. This output-oriented plant capacity notion compares the maximum value of outputs at the level of the current inputs to the maximum value of outputs when unlimited amounts of variable inputs are potentially available. Therefore, it determines how the maximal amount of efficient outputs is connected to the current amount of efficient outputs.

Kerstens, Sadeghi, and Van de Woestyne (2019b) recently argue and empirically illustrate that this output-oriented plant capacity utilization $PCU_o^{SR}(\mathbf{x}_p, \mathbf{x}_p^f, \mathbf{y}_p \mid \Lambda, \Gamma)$ is unrealistic in that the variable inputs amounts required to reach the maximum capacity outputs may simply be unavailable at either the firm or the industry level. This relates to what Johansen (1968) calls the attainability issue. Therefore, Kerstens, Sadeghi, and Van de Woestyne (2019b) define at the firm level a new attainable output-oriented plant capacity utilization as follows:

Definition 3.2. A short-run attainable output-oriented plant capacity utilization $APCU_o^{SR}$ at attainability level $\bar{\lambda} \in \mathbb{R}_+$ is defined by:

$$APCU_o^{SR}(\mathbf{x}_p, \mathbf{x}_p^f, \mathbf{y}_p, \bar{\lambda} \mid \Lambda, \Gamma) = \frac{DF_o(\mathbf{x}_p, \mathbf{y}_p \mid \Lambda, \Gamma)}{ADF_o^f(\mathbf{x}_p^f, \mathbf{y}_p, \bar{\lambda} \mid \Lambda, \Gamma)},$$

where the attainable output-oriented efficiency measure $ADF_o^f(\mathbf{x}_p^f, \mathbf{y}_p, \bar{\lambda} \mid \Lambda, \Gamma)$ at a certain attainability level $\bar{\lambda} \in \mathbb{R}_+$ is defined by:

$$ADF_o^f(\mathbf{x}_p^f, \mathbf{y}_p, \bar{\lambda} \mid \Lambda, \Gamma) \quad = \quad \max\{\varphi \mid \varphi \geq 0, 0 \leq \theta \leq \bar{\lambda}, \varphi \mathbf{y}_p \in P_{\Lambda,\Gamma}^f(\mathbf{x}_p^f, \theta \mathbf{x}_p^v)\} \tag{18}$$

Again, for $\bar{\lambda} \geq 1$, since $1 \leq DF_o(\mathbf{x}_p, \mathbf{y}_p \mid \Lambda, \Gamma) \leq ADF_o^f(\mathbf{x}_p^f, \mathbf{y}_p, \bar{\lambda} \mid \Lambda, \Gamma)$, notice that $0 < APCU_o(\mathbf{x}_p, \mathbf{x}_p^f, \mathbf{y}_p, \bar{\lambda} \mid \Lambda, \Gamma) \leq 1$. Also, for $\bar{\lambda} < 1$, since $1 \leq ADF_o^f(\mathbf{x}_p^f, \mathbf{y}_p, \bar{\lambda} \mid \Lambda, \Gamma) \leq DF_o(\mathbf{x}_p, \mathbf{y}_p \mid \Lambda, \Gamma)$, notice that $1 \leq APCU_o(\mathbf{x}_p, \mathbf{x}_p^f, \mathbf{y}_p, \bar{\lambda} \mid \Lambda, \Gamma)$. Moreover, in this case based on Proposition 2.2 we have $APCU_o(\mathbf{x}_p, \mathbf{x}_p^f, \mathbf{y}_p, \bar{\lambda} \mid \Lambda, \Gamma) < \infty$. As a result, we have the following remark.

Remark 3.2. *The attainable output-oriented plant capacity utilization* $APCU_o^{SR}(\boldsymbol{x}_p, \boldsymbol{x}_p^f, \boldsymbol{y}_p, \bar{\lambda} \mid \Lambda, \Gamma)$ *exists at the firm level under both the VRS and CRS cases as well as under both the convex and nonconvex assumptions.*

Moreover, the same authors remark that when experts cannot determine a plausible value for $\bar{\lambda}$, then one can opt for the input-oriented plant capacity measure below that is spared from this attainability issue. Based on the attainable output-oriented plant capacity utilization, one compares the maximal outputs at the level of observed inputs with the maximal outputs when variable inputs are scaled by $\bar{\lambda}$. Therefore, it clarifies how the current value of efficient outputs is connected to the maximal possible values of efficient outputs conditioned by the $\bar{\lambda}$ scalar.

Cesaroni, Kerstens, and Van de Woestyne (2017) introduce an input-oriented plant capacity measure under the VRS assumption using a couple of input-oriented efficiency measures.

Definition 3.3. The VRS short-run input-oriented plant capacity utilization (PCU_i^{SR}) is defined as follows:

$$PCU_i^{SR}(\mathbf{x}_p, \mathbf{x}_p^f, \mathbf{y}_p \mid \Lambda, VRS) = \frac{DF_i^{SR}(\mathbf{x}_p^f, \mathbf{x}_p^v, \mathbf{y}_p \mid \Lambda, VRS)}{DF_i^{SR}(\mathbf{x}_p^f, \mathbf{x}_p^v, \mathbf{0} \mid \Lambda, VRS)}, \tag{19}$$

where $DF_i^{SR}(\mathbf{x}_p^f, \mathbf{x}_p^v, \mathbf{y}_p \mid \Lambda, VRS)$ and $DF_i^{SR}(\mathbf{x}_p^f, \mathbf{x}_p^v, \mathbf{0} \mid \Lambda, VRS)$ are the sub-vector input efficiency measures defined in (13) and (15), respectively.

Remark 3.3. *Since $DF_i^{SR}(\boldsymbol{x}_p^f, \boldsymbol{x}_p^v, \boldsymbol{0} \mid \Lambda, VRS) \leq DF_i^{SR}(\boldsymbol{x}_p^f, \boldsymbol{x}_p^v, \boldsymbol{y}_p \mid \Lambda, VRS)$, notice that $1 \leq PCU_i^{SR}(\boldsymbol{x}, \boldsymbol{x}_p^f, \boldsymbol{y}_p \mid \Lambda, VRS)$. Thus, input-oriented plant capacity utilization has a lower limit of unity. Moreover, based on Proposition 2.7(i), we have $DF_i^{SR}(\boldsymbol{x}_p^f, \boldsymbol{x}_p^v, \boldsymbol{0} \mid \Lambda, VRS) > 0$. Therefore $PCU_i^{SR}(\boldsymbol{x}_p, \boldsymbol{x}_p^f, \boldsymbol{y}_p \mid \Lambda, VRS) < \infty$. As a result, the input-oriented plant capacity always exists at the firm level under VRS and under both the convex and nonconvex assumptions.*

Note that based on Proposition 2.7, we have $DF_i^{SR}(\mathbf{x}_p^f, \mathbf{x}_p^v, \mathbf{0} \mid \Lambda, CRS) = 0$. Hence, Definition 3.3 is invalid under the CRS case. Following Cesaroni, Kerstens, and Van de Woestyne (2017), we define an input-oriented plant capacity notion under CRS using a couple of input-oriented efficiency measures as follows:

Definition 3.4. The CRS short-run input-oriented plant capacity utilization (PCU_i^{SR}) can be defined as follows:

$$PCU_i^{SR}(\mathbf{x}_p, \mathbf{x}_p^f, \mathbf{y}_p, \mathbf{y}^\epsilon \mid \Lambda, CRS) = \frac{DF_i^{SR}(\mathbf{x}_p^f, \mathbf{x}_p^v, \mathbf{y}_p \mid \Lambda, CRS)}{DF_i^{SR}(\mathbf{x}_p^f, \mathbf{x}_p^v, \mathbf{y}^\epsilon \mid \Lambda, CRS)}, \tag{20}$$

where $DF_i^{SR}(\mathbf{x}_p^f, \mathbf{x}_p^v, \mathbf{y}_p \mid \Lambda, CRS)$ and $DF_i^{SR}(\mathbf{x}_p^f, \mathbf{x}_p^v, \mathbf{y}^\epsilon \mid \Lambda, CRS)$ are the sub-vector input efficiency measures at the current observed output level and at the $\mathbf{y}^\epsilon$ level, respectively.

Remark 3.4. *Since $DF_i^{SR}(\boldsymbol{x}_p^f, \boldsymbol{x}_p^v, \boldsymbol{y}^\epsilon \mid \Lambda, CRS) \leq DF_i^{SR}(\boldsymbol{x}_p^f, \boldsymbol{x}_p^v, \boldsymbol{y}_p \mid \Lambda, CRS)$, notice that $1 \leq PCU_i^{SR}(\boldsymbol{x}_p, \boldsymbol{x}_p^f, \boldsymbol{y}_p, \boldsymbol{y}^\epsilon \mid \Lambda, CRS)$. Thus, short-run input-oriented plant capacity utilization under the CRS has a lower limit of unity. Moreover, based on Proposition 2.9(ii), we have $DF_i^{SR}(\boldsymbol{x}_p^f, \boldsymbol{x}_p^v, \boldsymbol{y}^\epsilon \mid \Lambda, CRS) > 0$. Therefore, $PCU_i^{SR}(\boldsymbol{x}_p, \boldsymbol{x}_p^f, \boldsymbol{y}_p \mid \Lambda, CRS) < \infty$. As a result, input-oriented plant capacity in Definition 3.4 always exists at the firm level under CRS and under both the convex and nonconvex assumptions.*

Note that based on Proposition 2.9, we have $DF_i^{SR}(\mathbf{x}_p^f, \mathbf{x}_p^v, \mathbf{0} \mid \Lambda, VRS) = DF_i^{SR}(\mathbf{x}_p^f, \mathbf{x}_p^v, \mathbf{y}^\epsilon \mid \Lambda, VRS)$. Hence, a general form of the input-oriented plant capacity which is valid under both VRS and CRS cases can be defined as follows:

$$PCU_i^{SR}(\mathbf{x}_p, \mathbf{x}_p^f, \mathbf{y}_p, \mathbf{y}^\epsilon \mid \Lambda, \Gamma) = \frac{DF_i^{SR}(\mathbf{x}_p^f, \mathbf{x}_p^v, \mathbf{y}_p \mid \Lambda, \Gamma)}{DF_i^{SR}(\mathbf{x}_p^f, \mathbf{x}_p^v, \mathbf{y}^\epsilon \mid \Lambda, \Gamma)}. \tag{21}$$

3.2 Long-Run Plant Capacity Concepts

Cesaroni, Kerstens, and Van de Woestyne (2019) define long-run output- and input-oriented plant capacity concepts under the VRS assumption. In this subsection, we extend the long-run output- and input-oriented plant capacity concepts to the CRS case. Furthermore, we define a new long-run attainable output-oriented plant capacity concept: this has never been discussed in the literature.

Definition 3.5. The VRS long-run output-oriented plant capacity utilization (PCU_o^{LR}) is defined as:

$$PCU_o^{LR}(\mathbf{x}_p, \mathbf{y}_p \mid \Lambda, VRS) = \frac{DF_o(\mathbf{x}_p, \mathbf{y}_p \mid \Lambda, VRS)}{DF_o(\mathbf{y}_p \mid P_{\Lambda, VRS})}, \tag{22}$$

where $DF_o(\mathbf{x}_p, \mathbf{y}_p \mid \Lambda, VRS)$ and $DF_o(\mathbf{y}_p \mid P_{\Lambda,VRS})$ are output efficiency measures relative to technologies including all inputs respectively excluding all inputs.

Remark 3.5. *Notice that* $0 < PCU_o^{LR}(\boldsymbol{x}_p, \boldsymbol{y}_p \mid \Lambda, VRS) \leq 1$*, since* $1 \leq DF_o(\boldsymbol{x}_p, \boldsymbol{y}_p \mid \Lambda, VRS) \leq DF_o(\boldsymbol{y}_p \mid P_{\Lambda,VRS})$*. Thus, long-run output-oriented plant capacity utilisation has an upper limit of unity, but it has no lower limit. As a result, the long-run output-oriented plant capacity utilization always exists at the firm level under VRS and under both the convex and nonconvex assumptions.*

Note that based on Remark 2.2, we have $DF_o(\mathbf{y}_p \mid P_{CRS,\Gamma}) = \infty$. Hence, Definition 3.5 is invalid under the CRS case. Therefore, we define a long-run output-oriented plant capacity measures under the CRS assumption using a pair of output-oriented efficiency measures as follows:

Definition 3.6. The CRS long-run output-oriented plant capacity utilization (PCU_o^{LR}) is defined as:

$$PCU_o^{LR}(\mathbf{x}_p, \mathbf{y}_p, \mathbf{x}^\epsilon \mid \Lambda, CRS) = \frac{DF_o(\mathbf{x}_p, \mathbf{y}_p \mid \Lambda, CRS)}{DF_o(\mathbf{y}_p \mid P^{\mathbf{x}^\epsilon}_{\Lambda,CRS})}, \tag{23}$$

where $DF_o(\mathbf{x}_p, \mathbf{y}_p \mid \Lambda, VRS)$ and $DF_o(\mathbf{y}_p \mid P^{\mathbf{x}^\epsilon}_{CRS,\Gamma})$ are output efficiency measures relative to technologies including all inputs respectively excluding all inputs bigger or equal to x_ϵ.

Remark 3.6. *Notice that* $0 < PCU_o^{LR}(\boldsymbol{x}_p, \boldsymbol{y}_p, \boldsymbol{x}^\epsilon \mid \Lambda, CRS) \leq 1$*, since* $1 \leq DF_o(\boldsymbol{x}_p, \boldsymbol{y}_p \mid \Lambda, VRS) \leq DF_o(\boldsymbol{y}_p \mid P^{\boldsymbol{x}^\epsilon}_{CRS,\Gamma})$*. Thus, similar to the VRS case, the CRS long-run output-oriented plant capacity utilisation has an upper limit of unity, but it has no lower limit. As a result, the long-run output-oriented plant capacity utilization* (23) *always exists at the firm level under CRS and under both the convex and nonconvex assumptions.*

Note that based on Proposition 2.6, we have $DF_o(\mathbf{y}_p \mid P_{VRS,\Gamma}) = DF_o(\mathbf{y}_p \mid P^{\mathbf{x}^\epsilon}_{VRS,\Gamma})$. Hence, a general form of the long-run output-oriented plant capacity which is valid under both VRS and CRS cases can be defined as follows:

$$PCU_o^{LR}(\mathbf{x}_p, \mathbf{y}_p, \mathbf{x}^\epsilon \mid \Lambda, \Gamma) = \frac{DF_o(\mathbf{x}_p, \mathbf{y}_p \mid \Lambda, \Gamma)}{DF_o(\mathbf{y}_p \mid P^{\mathbf{x}^\epsilon}_{\Lambda,\Gamma})}. \tag{24}$$

In line with the short-run attainable output-oriented plant capacity notion in Definition 3.2 discussed above, we can now define a new long-run attainable output-oriented plant capacity utilization notion at the firm level as follows:

Definition 3.7. A long-run attainable output-oriented plant capacity utilization ($APCU_o^{LR}$) at attainability level $\bar{\lambda} \in \mathbb{R}_+$ is

$$APCU_o^{LR}(\mathbf{x}_p, \mathbf{y}_p, \bar{\lambda} \mid \Lambda, \Gamma) = \frac{DF_o(\mathbf{x}_p, \mathbf{y}_p \mid \Lambda, \Gamma)}{ADF_o(\mathbf{x}_p, \mathbf{y}_p, \bar{\lambda} \mid \Lambda, \Gamma)},$$

with $DF_o(\mathbf{x}_p, \mathbf{y}_p \mid \Lambda, \Gamma)$ as defined previously in (8) and where the long-run attainable output-oriented efficiency measure $ADF_o(\mathbf{x}_p, \mathbf{y}_p, \bar{\lambda} \mid \Lambda, \Gamma)$ at a certain attainability level $\bar{\lambda} \in \mathbb{R}_+$ is defined by:

$$ADF_o(\mathbf{x}_p, \mathbf{y}_p, \bar{\lambda} \mid \Lambda, \Gamma) = \max\{\varphi \mid \varphi \geq 0, 0 \leq \theta \leq \bar{\lambda}, \varphi \mathbf{y}_p \in P_{\Lambda,\Gamma}(\theta \mathbf{x}_p)\}. \tag{25}$$

For $\bar{\lambda} \geq 1$, since $1 \leq DF_o(\mathbf{x}_p, \mathbf{y}_p \mid \Lambda, \Gamma) \leq ADF_o(\mathbf{x}_p, \mathbf{y}_p, \bar{\lambda} \mid \Lambda, \Gamma)$, notice that $0 < APCU_o^{LR}(\mathbf{x}_p, \mathbf{y}_p, \bar{\lambda} \mid \Lambda, \Gamma) \leq 1$. Also, for $\bar{\lambda} < 1$, since $1 \leq ADF_o(\mathbf{x}_p, \mathbf{y}_p, \bar{\lambda} \mid \Lambda, \Gamma) \leq DF_o(\mathbf{x}_p, \mathbf{y}_p \mid \Lambda, \Gamma)$, notice that $1 \leq APCU_o^{LR}(\mathbf{x}_p, \mathbf{y}_p, \bar{\lambda} \mid \Lambda, \Gamma)$.

Remark 3.7. *Note that Propositions 1, 2 and 3 of Kerstens, Sadeghi, and Van de Woestyne (2019b) can be equally applied to the long-run attainable output-oriented plant capacity notion.*

Definition 3.8. The long-run input-oriented plant capacity utilization (PCU_i^{LR}) under VRS is defined as:

$$PCU_i^{LR}(\mathbf{x}_p, \mathbf{y}_p \mid \Lambda, VRS) = \frac{DF_i(\mathbf{x}_p, \mathbf{y}_p \mid \Lambda, VRS)}{DF_i(\mathbf{x}_p, \mathbf{0} \mid \Lambda, VRS)}, \tag{26}$$

where $DF_i(\mathbf{x}_p, \mathbf{y}_p \mid \Lambda, \Gamma)$ and $DF_i(\mathbf{x}_p, \mathbf{0} \mid \Lambda, \Gamma)$ are both input efficiency measures aimed at reducing all input dimensions relative to the VRS technology, whereby the latter efficiency measure is evaluated at a zero output level.

Remark 3.8. *Since $DF_i(\boldsymbol{x}_p, \boldsymbol{y}_p \mid \Lambda, VRS) \geq DF_i(\boldsymbol{x}_p, \boldsymbol{0} \mid \Lambda, VRS)$, notice that $1 \leq PCU_i^{LR}(\boldsymbol{x}_p, \boldsymbol{y}_p \mid \Lambda, VRS)$. Thus, long-run input-oriented plant capacity utilisation has a lower limit of unity, but it has no upper limit. Moreover, based on Proposition 2.7(i), we have $DF_i(\boldsymbol{x}_p, \boldsymbol{0} \mid \Lambda, VRS) > 0$. Therefore, $PCU_i^{LR}(\boldsymbol{x}_p, \boldsymbol{x}_p^f, \boldsymbol{y}_p \mid \Lambda, VRS) < \infty$. As a result, the long-run input-oriented plant capacity utilisation always exists at the firm level under VRS.*

Note that based on Proposition 2.7, we have $DF_i(\mathbf{x}_p, \mathbf{0} \mid \Lambda, CRS) = 0$. Hence, Definition 3.8 is invalid under the CRS case. We define a long-run input-oriented plant capacity measure under the CRS assumption using a couple of input-oriented efficiency measures as follows:

Definition 3.9. The long-run input-oriented plant capacity utilization (PCU_i^{LR}) under CRS is defined as:

$$PCU_i^{LR}(\mathbf{x}_p, \mathbf{y}_p, \mathbf{y}^\epsilon \mid \Lambda, CRS) = \frac{DF_i(\mathbf{x}_p, \mathbf{y}_p \mid \Lambda, CRS)}{DF_i(\mathbf{x}_p, \mathbf{y}^\epsilon \mid \Lambda, CRS)}, \tag{27}$$

where $DF_i(\mathbf{x}_p, \mathbf{y}_p \mid \Lambda, CRS)$ and $DF_i(\mathbf{x}_p, \mathbf{y}^\epsilon \mid \Lambda, CRS)$ are the input efficiency measures at the current observed output level and at the $\mathbf{y}^\epsilon$ output level, respectively.

Remark 3.9. *Since* $DF_i(\boldsymbol{x}_p, \boldsymbol{y}^\epsilon \mid \Lambda, CRS) \leq DF_i(\boldsymbol{x}_p, \boldsymbol{y}_p \mid \Lambda, CRS)$*, notice that* $1 \leq PCU_i^{LR}(\boldsymbol{x}_p, \boldsymbol{y}_p \mid \Lambda, CRS)$*. Thus, long-run input-oriented plant capacity utilization under CRS has a lower limit of unity. Moreover, based on Proposition 2.8(ii) we have* $DF_i(\boldsymbol{x}_p, \boldsymbol{y}^\epsilon \mid \Lambda, CRS) > 0$*. Thus,* $PCU_i^{LR}(\boldsymbol{x}_p, \boldsymbol{y}_p \mid \Lambda, CRS) < \infty$*. As a result, the input-oriented plant capacity in Definition 3.9 always exists at the firm level under CRS.*

Based on Proposition 2.8, we have $DF_i(\mathbf{x}_p, \mathbf{0} \mid \Lambda, VRS) = DF_i(\mathbf{x}_p, \mathbf{y}^\epsilon \mid \Lambda, VRS)$. Hence, a general form of the long-run input-oriented plant capacity which is valid under both VRS and CRS cases can be defined as follows:

$$PCU_i^{LR}(\mathbf{x}_p, \mathbf{y}_p, \mathbf{y}^\epsilon \mid \Lambda, \Gamma) = \frac{DF_i(\mathbf{x}_p, \mathbf{y}_p \mid \Lambda, \Gamma)}{DF_i(\mathbf{x}_p, \mathbf{y}^\epsilon \mid \Lambda, \Gamma)}. \tag{28}$$

3.3 Existence of Plant Capacity Concepts at the Firm Level: Conclusions

Wrapping up, the question regarding the existence of solutions for the short-run as well as the long-run output-, attainable output-, and input-oriented plant capacity concepts at the firm level can be answered affirmatively under both the VRS and CRS cases as well as under both the convex and nonconvex assumptions. We maintain mild and common axioms on the nonparametric technologies to establish these existence results at the firm level.

However, while short-run and long-run output-oriented plant capacity utilization may well exist from a mathematical viewpoint, these concepts may not be attainable: the amounts of variable inputs required to reach the maximum capacity outputs may simply be unavailable at the firm level. Similarly, while solutions for the short-run and long-run attainable output-oriented plant capacity utilization may exist, these concepts may again not be attainable depending on whether the choice of an attainability level $\bar{\lambda}$ is compatible with the real amount of available variable inputs or not. There are no such reservations for the input-oriented plant capacity concept.

We now turn to the question of existence of plant capacity concepts at the industry level.

4 Plant Capacity Concepts at the Industry Level

Similar to the firm-level plant capacity concepts and the question of their existence, it is also possible to devise new short-run output-, attainable output-, and input-oriented plant capacity concepts at the industry level and to check for their eventual existence. Exactly the same existence question pertains to the corresponding long-run output-, attainable output-, and input-oriented plant capacity concepts at the industry level.

4.1 Industry Output-Oriented Plant Capacity

Following Proposition 2.1 the constraints on the variable inputs for the short-run output-oriented efficiency measure as formulated in (11) are redundant and can be removed from the short-run technology $T^f_{\Lambda,\Gamma}$ at the firm level. Therefore, the firms can always consume less or more of its variable inputs to reach the maximum outputs capacity level. But, at the industry level we cannot just remove these variable input constraints.

Indeed, it remains an open question whether there exists a solution for all firms when they reach simultaneously their individual short-run output-oriented maximum plant capacity such that they respect the overall observed variable inputs? In other words, is it possible that all firms reach their full capacity simultaneously while consuming at most the overall amount of observed variable inputs? To answer this question, we formulate the following system of equations:

$$
\begin{cases}
\sum\limits_{k=1}^{K} \delta z^p_k \mathbf{y}_k \geq DF^f_o(\mathbf{x}^f, \mathbf{y} \mid \Lambda, \Gamma)\mathbf{y}_p, & p = 1, \ldots, K, \\
\sum\limits_{k=1}^{K} \delta z^p_k \mathbf{x}^f_k \leq \mathbf{x}^f_p, & p = 1, \ldots, K, \\
\sum\limits_{k=1}^{K} \delta z^p_k \mathbf{x}^v_k \leq \bar{\mathbf{x}}^v_p, & p = 1, \ldots, K, \\
\sum\limits_{p=1}^{K} \bar{\mathbf{x}}^v_p \leq \sum\limits_{p=1}^{K} \mathbf{x}^v_p, & \\
\mathbf{z}^p = (z^p_1 \ldots, z^p_K) \in \Lambda, \delta \in \Gamma, & p = 1, \ldots, K, \\
\bar{\mathbf{x}}^v_p \geq \mathbf{0}, & p = 1, \ldots, K.
\end{cases}
\tag{29}
$$

where $DF^f_o(\mathbf{x}^f_p, \mathbf{y}_p \mid \Lambda, \Gamma)$ is the short-run output-oriented efficiency measure defined in (11). Note that $\bar{\mathbf{x}}^v_p$ is a decision variable and that $\mathbf{x}^v_p$ is the observed variable input for firm p. Note that formulation (29) is general and applies to both CRS and VRS and to both convex and nonconvex technologies. Based on (29), all firms want to simultaneously produce at their maximum capacity and make a trade-off among their variable inputs such that the sum of optimal variable inputs be equal or smaller than the aggregate observed variable inputs (i.e., $\sum_{p=1}^{K} \bar{\mathbf{x}}^v_p \leq \sum_{p=1}^{K} \mathbf{x}^v_p$). Note that we reason here in terms of aggregate observed variable inputs: it is equally possible to apply the same reasoning to any aggregate amount of variable inputs that one deems available to the industry.

Remark 4.1. *(i) If the industry system of equations* (29) *is feasible, then the output-oriented plant capacity exists at the industry level with the given current overall level of variable inputs.*
(ii) If the industry system of equations (29) *is infeasible, then the output-oriented plant capacity at the industry level does not exist given the current overall amount of variable inputs.*

Notice that Färe and Karagiannis (2017, Section 3.3) discuss in a single output context an aggregate output-oriented plant capacity utilization notion as a weighted sum of individual output-oriented plant capacity utilization concepts over all firms in an industry. They consider the weighted arithmetic average of individual plant capacity utilization indices with the weights being potential output shares, defined by projecting the observed output onto the frontier (their method can be generalised into the multiple outputs case when output prices are available). However, their result, in contrast to our analysis, assumes that there are no limits on the aggregate variable inputs at the industry level and that no reallocation of variable inputs occurs across constituent firms.

Note that if the industry system of equations (29) is infeasible, then we face two options: either the aggregate amount of variable inputs should scale up from the current level (i.e., we need to allocate additional variable inputs to the industry to restore feasibility), or all firms cannot reach simultaneously the maximum capacity level with respect to current overall observed amounts of variable inputs (i.e., some firms must settle for less than full capacity utilisation). We treat these two options sequentially.

First, we assume that there is a possibility at the industry level to obtain some additional variable inputs that can be allocated to the firms. The question arises at least how much additional variable inputs are needed such that all firms are simultaneously able to reach their maximum plant capacity? To answer this question, we formulate the following model:

$$
\begin{array}{lll}
U^I = \min\limits_{\theta,\mathbf{z}^p,\bar{\mathbf{x}}^v_p} & \theta & \\
s.t. & \sum\limits_{k=1}^{K} \delta z^p_k \mathbf{y}_k \geq DF^f_o(\mathbf{x}^f_p, \mathbf{y}_p \mid \Lambda, \Gamma)\mathbf{y}_p, & p = 1, \ldots, K, \\
 & \sum\limits_{k=1}^{K} \delta z^p_k \mathbf{x}^f{}_k \leq \mathbf{x}^f_p, & p = 1, \ldots, K, \\
 & \sum\limits_{k=1}^{K} \delta z^p_k \mathbf{x}^v_k \leq \bar{\mathbf{x}}^v_p, & p = 1, \ldots, K, \\
 & \sum\limits_{p=1}^{K} \bar{\mathbf{x}}^v_p \leq \theta \sum\limits_{p=1}^{K} \mathbf{x}^v_p, & \\
 & \mathbf{z}^p \in \Lambda, \delta \in \Gamma, & p = 1, \ldots, K, \\
 & \theta \geq 0, \bar{\mathbf{x}}^v_p \geq \mathbf{0}, & p = 1, \ldots, K.
\end{array}
\tag{30}
$$

Notice that U^I is interpretable as the minimal expansion of the amount of industry variable inputs needed to be able to produce the full plant capacity outputs for all firms simultaneously.

Proposition 4.1. *Model* (30) *is feasible and* $U^I \leq 1$ *if and only if the system of equations* (29) *is feasible.*

If $U^I \leq 1$, then all firms can reach their maximum capacities with at most the overall observed variable inputs. If $U^I > 1$, then we need to scale up the industry observed variable inputs by at least U^I such that all firms can reach their maximum capacities. If the existing or available industry variable inputs is at least equal to $U^I \sum_{p=1}^{K} \bar{\mathbf{x}}^v_p$, then the maximum capacity of all firms can be used at the industry level.

However, as already illustrated and discussed in Kerstens, Sadeghi, and Van de Woestyne (2019b), the value of U^I can be quite huge. Therefore, scaling the observed industry variable inputs by an amount U^I may not be attainable at the industry level (see also subsection 4.2).

Second, it remains an open question whether there exists a solution for all firms when they optimize their capacity simultaneously without additional variable inputs. We introduce the industry output-oriented efficiency measure as follows:

Definition 4.1. The short-run industry output-oriented efficiency measure (IDF_o^f) for observation $(\mathbf{x}_p, \mathbf{y}_p)$ is

$$IDF_o^f(\mathbf{x}_p^f, \mathbf{y}_p \mid \Lambda, \Gamma) = \varphi_p^*,$$

where φ_p^* is the optimum value of φ_p in the following model:

$$\begin{array}{lll}
\max\limits_{\varphi_p, \mathbf{z}^p, \bar{\mathbf{x}}_p^v} & \sum\limits_{p=1}^{K} \varphi_p & \\
s.t & \sum\limits_{k=1}^{K} \delta z_k^p \mathbf{y}_k \geq \varphi_p \mathbf{y}_p, & p = 1, \ldots, K, \\
& \sum\limits_{k=1}^{K} \delta z_k^p \mathbf{x}_k^f \leq \mathbf{x}_p^f, & p = 1, \ldots, K, \\
& \sum\limits_{k=1}^{K} \delta z_k^p \mathbf{x}_k^v \leq \bar{\mathbf{x}}_p^v, & p = 1, \ldots, K, \\
& \sum\limits_{p=1}^{K} \bar{\mathbf{x}}_p^v \leq \sum\limits_{p=1}^{K} \mathbf{x}_p^v, & \\
& \mathbf{z}^p \in \Lambda, \delta \in \Gamma, & p = 1, \ldots, K, \\
& \varphi_p \geq 0, \bar{\mathbf{x}}_p^v \geq \mathbf{0}, & p = 1, \ldots, K.
\end{array} \tag{31}$$

where Λ and Γ allow for both convex and nonconvex technologies and both CRS and VRS technologies, respectively. Industry model (31) is a central resource allocation model including K linear programs corresponding to each firm with a bogus objective function and a common constraint on the overall observed variable inputs in the industry. Specifically, it aims to maximise the output-oriented plant capacity of all firms by reallocating the variable inputs such that the overall observed amount of variable inputs is satisfied.

Let φ_p^{**} be the optimum value of φ_p in industry model (31) without its last functional constraint $\sum_{p=1}^{K} \bar{\mathbf{x}}_p^v \leq \sum_{p=1}^{K} \mathbf{x}_p^v$. In this case, we obtain: $DF_o^f(\mathbf{x}_p^f, \mathbf{y}_p \mid \Lambda, \Gamma) = \varphi_p^{**}$. Consequently, by ignoring this global industry constraint of model (31), both the industry and firm level output-oriented efficiency measures $IDF_o^f(\mathbf{x}_p^f, \mathbf{y}_p \mid \Lambda, \Gamma)$ and $DF_o^f(\mathbf{x}_p^f, \mathbf{y}_p \mid \Lambda, \Gamma)$ coincide: $IDF_o^f(\mathbf{x}_p^f, \mathbf{y}_p \mid \Lambda, \Gamma) = DF_o^f(\mathbf{x}_p^f, \mathbf{y}_p \mid \Lambda, \Gamma) = \varphi_p^{**}$. It is important to note that -to the best of our knowledge- we are the first to address the concept of an industry level output-oriented plant capacity.

Using the short-run industry output-oriented efficiency measure (Definition 4.1), one can define the short-run industry output-oriented plant capacity utilization as follows:

Definition 4.2. The short-run industry output-oriented plant capacity utilization ($IPCU_o^{SR}$) for observation $(\mathbf{x}_p, \mathbf{y}_p)$ is

$$IPCU_o^{SR}(\mathbf{x}_p, \mathbf{x}_p^f, \mathbf{y}_p \mid \Lambda, \Gamma) = \frac{DF_o(\mathbf{x}_p, \mathbf{y}_p \mid \Lambda, \Gamma)}{IDF_o^f(\mathbf{x}_p^f, \mathbf{y}_p \mid \Lambda, \Gamma)}.$$

Because $IDF_o^f(\mathbf{x}_p^f, \mathbf{y}_p \mid \Lambda, \Gamma) \leq DF_o^f(\mathbf{x}_p^f, \mathbf{y}_p \mid \Lambda, \Gamma)$, $IPCU_o^{SR}(\mathbf{x}_p, \mathbf{x}_p^f, \mathbf{y}_p \mid \Lambda, \Gamma) \geq PCU_o^{SR}(\mathbf{x}_p, \mathbf{x}_p^f, \mathbf{y}_p \mid \Lambda, \Gamma)$. Therefore, the short-run industry output-oriented measure of plant capacity utilization is larger than or equal to the traditional measure of short-run output-oriented plant capacity utilization. By analogy, we can distinguish between the short-run industry biased plant capacity measure $IDF_o^f(\mathbf{x}_p^f, \mathbf{y}_p \mid \Lambda, \Gamma)$ and the short-run industry unbiased plant capacity measure $IPCU_o^{SR}(\mathbf{x}_p, \mathbf{x}_p^f, \mathbf{y}_p \mid \Lambda, \Gamma)$, where the ratio of efficiency measures ensures elimination of any existing inefficiency.

Observe that there is no a priori relation between both the biased and unbiased versions of the short-run output-oriented measures of plant capacity utilization at the firm and industry levels. Thus, we can write $IDF_o^f(\mathbf{x}_p^f, \mathbf{y}_p, \bar{\lambda}) \gtreqless DF_o^f(\mathbf{x}_p^f, \mathbf{y}_p, \bar{\lambda})$ and $IPCU_o^{SR}(\mathbf{x}_p, \mathbf{x}_p^f, \mathbf{y}_p, \bar{\lambda}) \gtreqless PCU_o^{SR}(\mathbf{x}_p, \mathbf{x}_p^f, \mathbf{y}_p, \bar{\lambda})$.

Remark 4.2. *Note that the same structure as developed in this subsection can be used to define the long-run output-oriented plant capacity at the industry level. Since we have no partitioning of the inputs in this long-run case, hence $N_f = 0$ and $N = N_v$. Therefore, removing the constraints corresponding to the fixed inputs from the system of equations* (29) *and in models* (30) *and* (31)*, and furthermore replacing $DF_o^f(\boldsymbol{x}_p^f, \boldsymbol{y}_p \mid \Lambda, \Gamma)$ with $DF_o(\boldsymbol{y}_p \mid P_{\Lambda,\Gamma}^{\boldsymbol{x}^\epsilon})$ in the system of equations* (29) *and in model* (30) *leads to the corresponding concepts for the long-run industry output-oriented plant capacity. As a result, Proposition 4.1 as well as Definitions 4.1 and 4.2 can be defined for the long-run output-oriented plant capacity.*

The real risk that the output-oriented plant capacity does not exist at the industry level provides a motivation for considering the attainable output-oriented plant capacity notion developed by Kerstens, Sadeghi, and Van de Woestyne (2019b). We now turn to the question whether it does any better in terms of existence at the industry level.

4.2 Industry Attainable Output-Oriented Plant Capacity

There are sometimes additional variable inputs to allocate to the firms. As mentioned above, if the available additional variable inputs are at least as much as $U^I \sum_{p=1}^{K} \bar{\mathbf{x}}_p^v$ where U^I is the optimal value of model (30) and $\bar{\mathbf{x}}_p^v$ represents the observed variable inputs of firm p, then we can allocate the available additional inputs to the firms such that all firms reach their full capacity.

However, consider the situation where the available variable resources are smaller than the minimum level which is needed to reach full capacity in all firms (i.e., the system (29) is infeasible). In this case, Kerstens, Sadeghi, and Van de Woestyne (2019b) define the industry attainable output-oriented plant capacity under the VRS assumption solely. A generalised definition of the industry attainable output-oriented efficiency measure can now be defined as follows:

Definition 4.3. The short-run industry attainable output-oriented efficiency measure ($IADF_o^f$) for observation $(\mathbf{x}_p, \mathbf{y}_p)$ is

$$IADF_o^f(\mathbf{x}_p^f, \mathbf{y}_p \mid \Lambda, \Gamma) = \varphi_p^*,$$

where φ_p^* is the optimum value of φ_p in the following model:

$$\begin{array}{lll}
\max\limits_{\varphi_p, \mathbf{z}^p, \bar{\mathbf{x}}_p^v} & \sum\limits_{p=1}^{K} \varphi_p & \\
s.t & \sum\limits_{k=1}^{K} \delta z_k^p \mathbf{y}_k \geq \varphi_p \mathbf{y}_p, & p = 1, \dots, K, \\
& \sum\limits_{k=1}^{K} \delta z_k^p \mathbf{x}_k^f \leq \mathbf{x}_p^f, & p = 1, \dots, K, \\
& \sum\limits_{k=1}^{K} \delta z_k^p \mathbf{x}_k^v \leq \bar{\mathbf{x}}_p^v, & p = 1, \dots, K, \\
& \sum\limits_{p=1}^{K} \bar{\mathbf{x}}_p^v \leq \bar{\lambda} \sum\limits_{p=1}^{K} \bar{\mathbf{x}}_p^v, & \\
& \mathbf{z}^p \in \Lambda, \delta \in \Gamma, & p = 1, \dots, K, \\
& \varphi_p \geq 0, \bar{\mathbf{x}}_p^v \geq \mathbf{0}, & p = 1, \dots, K.
\end{array} \tag{32}$$

where Λ and Γ allow for both convex and nonconvex technologies and both CRS and VRS technologies, respectively. The constraint $\sum_{p=1}^{K} \bar{\mathbf{x}}_p^v \leq \bar{\lambda} \sum_{p=1}^{K} \mathbf{x}_p^v$ shows that the sum of the decision variables $\bar{\mathbf{x}}_p^v$ cannot be higher than the attainable amount of total variable inputs at the industry level.

Using the short-run industry attainable output-oriented efficiency measure of Definition 4.3, the short-run industry attainable output-oriented plant capacity utilization is defined as follows:

Definition 4.4. The short-run industry attainable output-oriented plant capacity utilization ($IAPCU_o^{SR}$) at attainability level $\bar{\lambda} \in \mathbb{R}_+$ for observation $(\mathbf{x}_p, \mathbf{x}_p)$ is

$$IAPCU_o^{SR}(\mathbf{x}_p, \mathbf{x}_p^f, \mathbf{y}_p, \bar{\lambda}) = \frac{DF_o(\mathbf{x}_p, \mathbf{y}_p)}{IADF_o^f(\mathbf{x}_p^f, \mathbf{y}_p, \bar{\lambda})}. \tag{33}$$

Note that $IAPCU_o^{SR}(\mathbf{x}_p, \mathbf{x}_p^f, \mathbf{y}_p, \bar{\lambda}) \geq PCU_o^{SR}(\mathbf{x}_p, \mathbf{x}_p^f, \mathbf{y}_p)$ since their denominators are ranked as follows: $IADF_o^f(\mathbf{x}_p^f, \mathbf{y}_p, \bar{\lambda}) \leq DF_o^f(\mathbf{x}_p^f, \mathbf{y}_p)$. Therefore, the short-run industry attainable output-oriented plant capacity measure is always larger than or equal to the short-run output-oriented plant capacity measure. By analogy, one may differentiate between the short-run industry attainable unbiased plant capacity measure $IAPCU_o^{SR}(\mathbf{x}_p, \mathbf{x}_p^f, \mathbf{y}_p, \bar{\lambda})$ and the short-run industry attainable biased plant capacity measure $IADF_o^f(\mathbf{x}_p^f, \mathbf{y}_p, \bar{\lambda})$, whereby the ratio of efficiency measures guarantees removing any existing inefficiency in the former.

Note that there is no determinate relation between both the biased and unbiased versions of the short-run attainable output-oriented measures of plant capacity utilization at the firm and industry levels. Thus, we obtain $IADF_o^f(\mathbf{x}_p^f, \mathbf{y}_p, \bar{\lambda}) \gtreqless ADF_o^f(\mathbf{x}_p^f, \mathbf{y}_p, \bar{\lambda})$ and $IAPCU_o^{SR}(\mathbf{x}_p, \mathbf{x}_p^f, \mathbf{y}_p, \bar{\lambda}) \gtreqless APCU_o^{SR}(\mathbf{x}_p, \mathbf{x}_p^f, \mathbf{y}_p, \bar{\lambda})$.

The attainability level $\bar{\lambda}$ in model (32) can be varied in a subinterval of $(0, \infty)$. To discover the feasible area for $\bar{\lambda}$, Kerstens, Sadeghi, and Van de Woestyne (2019b) define the critical point L^I solely for the VRS case. Here, we formulate a more general model to determine this critical point L^I as follows:

$$
\begin{aligned}
L^I = \quad & \min_{\theta, \mathbf{z}^p, \bar{\mathbf{x}}_p^v} \quad \theta \\
& s.t \quad \sum_{k=1}^{K} \delta z_k^p \mathbf{x}_k^f \leq \mathbf{x}_p^f, \qquad p = 1, \ldots, K, \\
& \qquad \sum_{k=1}^{K} \delta z_k^p \mathbf{x}_k^v \leq \bar{\mathbf{x}}_p^v, \qquad p = 1, \ldots, K, \\
& \qquad \sum_{p=1}^{K} \bar{\mathbf{x}}_p^v \leq \theta \sum_{p=1}^{K} \mathbf{x}_p^v, \\
& \qquad \mathbf{z}^p \in \Lambda, \delta \in \Gamma, \qquad p = 1, \ldots, K, \\
& \qquad \theta \geq 0, \bar{\mathbf{x}}_p^v \geq \mathbf{0}, \qquad p = 1, \ldots, K.
\end{aligned} \tag{34}
$$

Similar to Proposition 4 of Kerstens, Sadeghi, and Van de Woestyne (2019b, p. 1142), we now have the following proposition:

Proposition 4.2. *Industry model* (32) *is feasible if and only if* $\bar{\lambda} \geq L^I$.

Note that Proposition 4 of Kerstens, Sadeghi, and Van de Woestyne (2019b, p. 1142) contains some further details as to the existence of solutions with regard to another critical upper bound U^I (see (30)). Thus, under mild conditions on $\bar{\lambda}$ the attainable output-oriented plant capacity does exist at the industry level.

Remark 4.3. *Note that the same structure as developed in this subsection can be used to define the long-run attainable output-oriented plant capacity at the industry level. Since we have no partitioning for the inputs in the long-run case, hence* $N_f = 0$ *and* $N = N_v$. *Therefore, removing the constraints corresponding to the fixed inputs from industry model* (32) *leads to the corresponding model of the long-run industry attainable output-oriented plant capacity. As a result, Definitions 4.3 and 4.4 and Proposition 4.2 can be developed for the long-run attainable output-oriented plant capacity.*

4.3 Industry Input-Oriented Plant Capacity

There are constraints on the variable inputs for the short-run input-oriented efficiency measure at the firm level as formulated in (17). Therefore, the firms can always consume less or an equal amount of its variable inputs to reach the minimal outputs $\mathbf{y}^{\epsilon}$ defining the input-oriented plant capacity level. While Proposition 2.3 allows to remove the output constraints at the firm level, these same constraints cannot be removed at the industry level.

However, it remains an open question whether there exists a solution for all firms when they reach simultaneously their individual short-run input-oriented plant capacity such that they respect the overall amount of observed variable inputs? To answer this question, we formulate the following system of equations:

$$\begin{cases} \sum_{k=1}^{K} \delta z_k^p \mathbf{y}_k \geq \mathbf{y}^{\epsilon}, & p = 1, \dots, K, \\ \sum_{k=1}^{K} \delta z_k^p \mathbf{x}_k^f \leq \mathbf{x}_p^f, & p = 1, \dots, K, \\ \sum_{k=1}^{K} \delta z_k^p \mathbf{x}_k^v \leq DF_i^{SR}(\mathbf{x}_p^f, \mathbf{x}_p^v, \mathbf{y}_p^{\epsilon} \mid \Lambda, \Gamma) \bar{\mathbf{x}}_p^v, & p = 1, \dots, K, \\ \sum_{p=1}^{K} \bar{\mathbf{x}}_p^v \leq \sum_{p=1}^{K} \mathbf{x}_p^v, & \\ \mathbf{z}^p \in \Lambda, \delta \in \Gamma, \bar{\mathbf{x}}_p^v \geq \mathbf{0} & p = 1, \dots, K, \end{cases} \tag{35}$$

where $DF_i^{SR}(\mathbf{x}_p^f, \mathbf{x}_p^v, \mathbf{y}_p^{\epsilon} \mid \Lambda, \Gamma)$ is the short-run input-oriented efficiency measure (17). Note that $\mathbf{x}_p^v$ is a decision variable and $\bar{\mathbf{x}}_p^v$ is the observed variable inputs of firm p. Note that formulation (35) is general: it applies to both CRS and VRS and to both convex and nonconvex technologies. Based on (35), all firms want to start working at their full input-oriented capacity simultaneously and make a trade-off among their variable inputs such that the sum of optimal variable inputs be equal or smaller than the aggregate observed variable inputs (i.e., $\sum_{p=1}^{K} \bar{\mathbf{x}}_p^v \leq \sum_{p=1}^{K} \mathbf{x}_p^v$).

Proposition 4.3. *The industry system of equations* (35) *is feasible.*

Based on Proposition 4.3, the industry system of equations (35) is always feasible. Hence, all firms can reach their full short-run input-oriented plant capacity by consuming the overall observed variable inputs. Since the industry system of equations (35) is always feasible, the short-run industry input-oriented capacity exists at the current level of industry variable inputs.

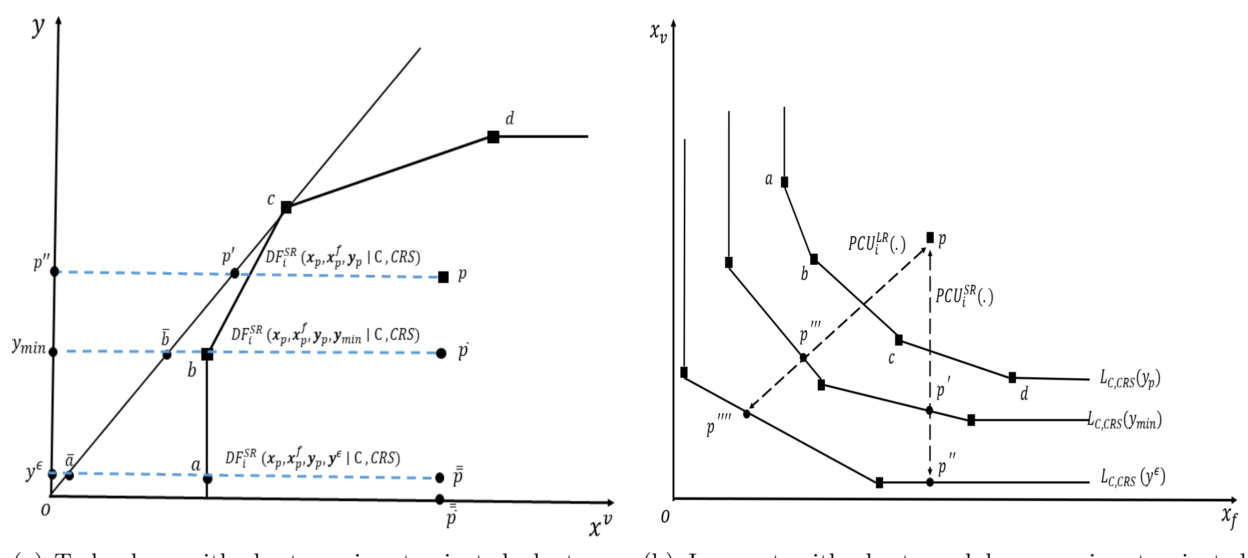
This result contrasts with the lack of definite results in subsection 4.1 on the existence of the traditional short-run output-oriented plant capacity concept at the industry level. It makes the short-run input-oriented capacity concept a valuable alternative to the traditional short-run output-oriented plant capacity notion.

Remark 4.4. *Note that the same structure as developed in this subsection can be used to define the long-run input-oriented plant capacity at the industry level. Since we have no partitioning* *for the inputs in the long-run case, hence $N_f = 0$ and $N = N_v$. Therefore, removing the constraints corresponding to the fixed inputs from industry system of equations* (35) *and replacing $DF_i^{SR}(\boldsymbol{x}_p^f, \boldsymbol{x}_p^v, \boldsymbol{y}^\epsilon \mid \Lambda, \Gamma)$ with $DF_i(\boldsymbol{x}_p, \boldsymbol{y}^\epsilon \mid \Lambda, \Gamma)$ leads to the corresponding result for the long-run industry input-oriented plant capacity. As a result, the industry long-run input-oriented plant capacity exists at the current level of industry inputs.*

4.4 Existence of Plant Capacity Concepts at the Industry Level: Conclusions

Wrapping up our results as to the existence of solutions for the industry problem, we can state the following. If the system of equations (29) is feasible, then the output-oriented plant capacity exists at the industry level with the given current overall level of variable inputs: we see that existence and attainability are intimately linked at the industry level. For the industry attainable output-oriented plant capacity, we have shown that industry model (32) is feasible if and only if the attainability level $\bar{\lambda}$ respects a critical parameter L^I. Finally, for the industry input-oriented plant capacity, we have shown that industry model (35) is always feasible.

5 Sensitivity of Plant Capacity Concepts for the Choice of $\mathbf{x}^\epsilon$ and $\mathbf{y}^\epsilon$



(a) Technology with short-run input oriented plant capacity measures.

(b) Isoquant with short- and long-run input oriented plant capacity measures.

Figure 1: Sensitivity of $PCU_i^{SR}(.)$ and $PCU_i^{LR}(.)$ for the choice of $\mathbf{y}^\epsilon$.

To explain our theoretical developments with regard to the short-run input-oriented plant capacity (21) we discuss Figure 1a. This two dimensional figure is drawn in variable input and output space.

It displays a convex VRS technology represented by the polyline *abcd* and the horizontal extension to the right of point d. It also displays a CRS cone starting at the origin and passing through point c. For an output level y_{min}, point $\overline{p}$ is projected on the cone at point $\overline{b}$. By contrast, for an output level y^{ϵ} point $\overline{\overline{p}}$ is projected on the cone at point $\overline{a}$ under the CRS case. The latter solution is the closest we can get to the origin of the cone.

Figure 1b develops the geometric intuition behind the short-run and long-run plant capacity measures under the CRS case. The isoquant denoting the combinations of fixed and variable inputs yielding a given output level $L_{C,CRS}(y_p)$ is represented by the polyline *abcd* and its vertical and horizontal extensions at a and d, respectively. We focus on observation p to illustrate first the short-run input-oriented plant capacity utilisation measure: for a given fixed input vector, it seems logical to look for a reduction in variable inputs for given fixed inputs towards the translated point p' that is situated outside the isoquant $L_{C,CRS}(y_p)$ because it produces an output vector y_{min} (it is compatible with the isoquant $L_{C,CRS}(y_{min})$ that is situated below the isoquant $L_{C,CRS}(y_p)$). It also seems logical to look for a reduction in variable inputs for given fixed inputs towards the translated point p'' that is situated outside the isoquant $L(y_p)$ because it produces an output vector y^{ϵ} (it is compatible with the isoquant $L_{C,CRS}(y^{\epsilon})$ that is situated below the isoquants $L_{C,CRS}(y_p)$ and $L_{C,CRS}(y_{min})$). Note that we have $PCU_i^{SR}(\mathbf{x}_p, \mathbf{x}_p^f, \mathbf{y}_p, \mathbf{y}_{min} \mid \Lambda, CRS) < PCU_i^{SR}(\mathbf{x}_p, \mathbf{x}_p^f, \mathbf{y}_p, \mathbf{y}^{\epsilon} \mid \Lambda, CRS)$ and if $\mathbf{y}^{\epsilon}$ becomes smaller and smaller, $PCU_i^{SR}(\mathbf{x}_p, \mathbf{x}_p^f, \mathbf{y}_p, \mathbf{y}^{\epsilon} \mid \Lambda, CRS)$ will become bigger and bigger (see Proposition 5.1(i) infra).

The long-run input-oriented plant capacity measure now equally looks for a reduction in all inputs towards the translated point p''' that is situated outside the isoquant $L(y_p)$ because it corresponds to an output level y_{min}. Also, it looks for a reduction in all inputs towards the translated point p'''' that is situated again outside the isoquant $L_{C,CRS}(y_p)$ because it corresponds to an output level y^{ϵ}. Note that we have $PCU_i^{LR}(\mathbf{x}_p, \mathbf{y}_p, \mathbf{y}_{min} \mid \Lambda, CRS) < PCU_i^{LR}(\mathbf{x}_p, \mathbf{y}_p, \mathbf{y}^{\epsilon} \mid \Lambda, CRS)$ and if $\mathbf{y}^{\epsilon}$ becomes smaller and smaller, $PCU_i^{LR}(\mathbf{x}_p, \mathbf{y}_p, \mathbf{y}^{\epsilon} \mid \Lambda, CRS)$ will become bigger and bigger (see Proposition 5.1(ii) infra).

While a solution for the short-run and long-run input-oriented plant capacities exist at both the firm and industry levels, there are numerical issues in the CRS case. Indeed, under the CRS case one can prove the following result for the short-run and long-run input-oriented plant capacities:

Proposition 5.1. *We have:*

$$\begin{aligned} &(i) \quad \lim_{\boldsymbol{y}^{\epsilon} \to \boldsymbol{0}} PCU_i^{SR}(\boldsymbol{x}_p, \boldsymbol{x}_p^f, \boldsymbol{y}_p, \boldsymbol{y}^{\epsilon} \mid \Lambda, CRS) = \infty, \\ &(ii) \quad \lim_{\boldsymbol{y}^{\epsilon} \to \boldsymbol{0}} PCU_i^{LR}(\boldsymbol{x}_p, \boldsymbol{y}_p, \boldsymbol{y}^{\epsilon} \mid \Lambda, CRS) = \infty. \end{aligned} \tag{36}$$

Thus, the smaller y^{ϵ} the more the short-run and long-run input-oriented plant capacities become arbitrarily large. This reveals that the above theoretical solution for the CRS case (21) and (28) may face numerical problems.

Obviously, there are rather straightforward solutions to this problem. For instance, if we consider $\mathbf{y}^{\epsilon} = \mathbf{y}_{min}$ in the CRS case, then we can define the $PCU_i^{SR}(\mathbf{x}_p, \mathbf{x}_p^f, \mathbf{y}_p, \mathbf{y}_{min} \mid \Lambda, \Gamma)$ and $PCU_i^{LR}(\mathbf{x}_p, \mathbf{y}_p, \mathbf{y}_{min} \mid \Lambda, CRS)$ under the CRS case, and we can obtain some more reasonable results. Actually, we have $PCU_i^{SR}(\mathbf{x}_p, \mathbf{x}_p^f, \mathbf{y}_p, \mathbf{y}_{min} \mid \Lambda, CRS) \neq PCU_i^{SR}(\mathbf{x}_p, \mathbf{x}_p^f, \mathbf{y}_p, \mathbf{y}_{min} \mid \Lambda, VRS)$ as well as $PCU_i^{LR}(\mathbf{x}_p, \mathbf{y}_p, \mathbf{y}_{min} \mid \Lambda, CRS) \neq PCU_i^{LR}(\mathbf{x}_p, \mathbf{y}_p, \mathbf{y}_{min} \mid \Lambda, VRS)$.

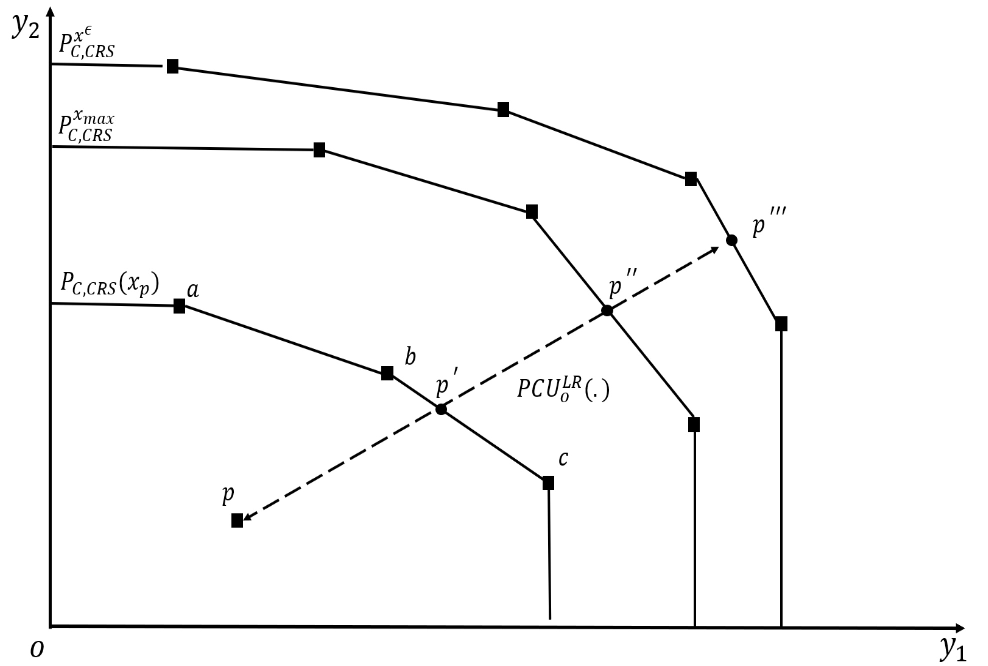


Figure 2: Isoquant with long-run output oriented plant capacity measure: sensitivity of $PCU_o^{LR}(.)$ for the choice of $\mathbf{x}^{\epsilon}$

Figure 2 develops the geometric intuition behind the long-run output-oriented plant capacity measure. The isoquant denoting the combinations of two outputs yielding a given input level $P_{C,CRS}(x_p)$ is represented by the polyline abc and its horizontal and vertical extensions at a and c, respectively. We focus on observation p to illustrate first the long-run output-oriented plant capacity utilisation measure. The long-run output-oriented plant capacity measure $PCU_o^{LR}(x_p, \mathbf{y}_p, x_{max} \mid \Lambda, CRS)$ -its corresponding isoquant is labeled $P_{C,CRS}^{x_{max}}$ in Figure 2- scales up all inputs at most as much as x_{max} to reach a translated point p'' that allows maximizing the vector of outputs. In a similar way, the long-run output-oriented plant capacity measure $PCU_o^{LR}(x_p, \mathbf{y}_p, x^{\epsilon} \mid \Lambda, CRS)$ -its corresponding isoquant is labeled $P_{C,CRS}^{x^{\epsilon}}$ in Figure 2- scales up all inputs at most as much as x^{ϵ} to reach a translated point p''' that allows maximizing the vector of outputs. Note that we have $PCU_o^{LR}(x_p, \mathbf{y}_p, x_{max} \mid \Lambda, CRS) > PCU_o^{LR}(x_p, \mathbf{y}_p, x^{\epsilon} \mid \Lambda, CRS)$ and if $\mathbf{x}^{\epsilon}$ becomes bigger and bigger, $PCU_o^{LR}(x_p, \mathbf{y}_p, x^{\epsilon} \mid \Lambda, CRS)$ will become smaller and smaller (see Proposition 5.2 infra).

In particular, under the CRS case one can prove the following result for the long-run output-oriented plant capacity:

Proposition 5.2. *We have:*

$$\lim_{\boldsymbol{x}^{\epsilon}\to\infty} PCU_o^{LR}(\boldsymbol{x}_p, \boldsymbol{y}_p, \boldsymbol{x}^{\epsilon} \mid \Lambda, CRS) = 0$$

Thus, the bigger $\mathbf{x}^{\epsilon}$ the more the long-run output-oriented plant capacities become arbitrarily small. This reveals that the above theoretical solution for the CRS case (23) may face numerical problems.

We end with establishing some relations between the long-run output-oriented plant capacity and the long-run attainable output-oriented plant capacity concepts:

Proposition 5.3. *We have:*

$$\begin{aligned} &(i) \quad \lim_{\bar{\lambda}\to\infty} APCU_o^{LR}(\boldsymbol{x}_p, \boldsymbol{y}_p, \bar{\lambda} \mid \Lambda, CRS) = PCU_o^{LR}(\boldsymbol{x}_p, \boldsymbol{y}_p \mid \Lambda, CRS) = 0, \\ &(ii) \quad \lim_{\bar{\lambda}\to\infty} APCU_o^{LR}(\boldsymbol{x}_p, \boldsymbol{y}_p, \bar{\lambda} \mid \Lambda, VRS) = PCU_o^{LR}(\boldsymbol{x}_p, \boldsymbol{y}_p \mid \Lambda, VRS). \end{aligned} \tag{37}$$

Both long-run output-oriented plant capacity concepts are related to one another when $\bar{\lambda}$ approaches ∞.

Remark 5.1. *Note that if we choose $\bar{\lambda}$ and $\boldsymbol{x}^{\epsilon}$ such that $\bar{\lambda}\boldsymbol{x}_p = \boldsymbol{x}^{\epsilon}$, then we have $ADF_o^f(\boldsymbol{x}_p, \boldsymbol{y}_p, \bar{\lambda} \mid \Lambda, \Gamma) = DF_o(\boldsymbol{y}_p \mid P_{\Lambda,\Gamma}^{\boldsymbol{x}^{\epsilon}})$. As a result, in this case we have $APCU_o^{LR}(\boldsymbol{x}_p, \boldsymbol{y}_p, \bar{\lambda} \mid \Lambda, \Gamma) = PCU_o^{LR}(\boldsymbol{x}_p, \boldsymbol{y}_p, \boldsymbol{x}^{\epsilon} \mid \Lambda, \Gamma)$.*

To conclude this discussion about the computational issues surrounding the above plant capacity concepts, it is good to consider the following argument. Despite the fact that the seminal contributions of Färe (1988), Färe, Grosskopf, and Kokkelenberg (1989) and Färe, Grosskopf, and Valdmanis (1989) define the output-oriented plant capacity concept with regard to a CRS technology, one must remember that CRS is unlikely a realistic assumption for any general technology. The CRS assumption implicitly presupposes the economy is in some form of Walrasian general equilibrium. Instead, we consider the more general VRS technology to be the true technology, while the CRS technology is just an auxiliary technology useful to determine, e.g., returns to scale for individual units.[3] Therefore, it is preferable to use the VRS assumption to compute any of the

[3] Scarf (1994, pp. 114–115) mocks the possibility of a CRS technology as follows: "Both linear programming and the Walrasian model of equilibrium make the fundamental assumption that the production possibility set displays constant or decreasing returns to scale; that there are no economies associated with production at a high scale. I find this an absurd assumption, contradicted by the most casual of observations. Taken literally, the assumption of constant returns to scale in production implies that if technical knowledge were universally available we could all trade only in factors of production, and assemble in our own backyards all of the manufactured goods whose services we would like to consume. If I want an automobile at a specified future date, I would purchase steel, glass, rubber, electrical wiring and tools, hire labor of a variety of skills on a part-time basis, and simply make the automobile myself. I would grow my own food, cut and sew my own clothing, build my own computer chips and assemble and disassemble my own international communication system whenever I need to make a telephone call, without any loss of efficiency. Notwithstanding the analysis offered by Adam Smith more than two centuries ago, I would manufacture pins as I needed them."

plant capacity notions. Computational problems related to some of the plant capacity notions for CRS technologies are probably minor issues of little relevance for empirical practice.

6 Conclusions

This contribution has first defined the short-run and long-run versions of the traditional output-oriented, the attainable output-oriented, and the input-oriented plant capacity notions. We have first established that all these plant capacity notions are well-defined at the firm level for general nonparametric technologies under both CRS and VRS assumptions and under both convexity and nonconvexity. This has led to some theoretical refinements in the definition of the input-oriented plant capacity notions with regard to a CRS technology and it has led to the definition of a new long-run attainable output-oriented plant capacity concept.

In addition, we have answered the question as to the existence of the same three plant capacity concepts at the industry level. First, we establish that the short-run output-oriented plant capacity notion may not exist at the industry level. This result is obviously connected to the attainability issue that triggered the introduction of the short-run attainable output-oriented plant capacity concept in the first place. Second, the short-run attainable output-oriented plant capacity exists for a proper choice of an attainability level $\bar{\lambda}$. Third, the short-run input-oriented plant capacity notion always exists at the industry level. Furthermore, these same industry results immediately transpose to the corresponding long-run plant capacity notions.

We also pay attention to the computational issues regarding the definition of the input-oriented plant capacity notion with regard to a CRS technology: these practical issues are less important than they appear if one realises that the true technology is a VRS technology. In conclusion, this contribution casts some doubt on the widespread use of the traditional output-oriented plant capacity concept. As an alternative one may use either the attainable output-oriented plant capacity if one is able to specify some valid attainability level $\bar{\lambda}$, or the input-oriented plant capacity notion.

Just to sketch one avenue for future work, following Färe and Karagiannis (2017) it could be interesting to investigate the conditions under which aggregate plant capacity utilization concepts can be defined.

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Appendices: Supplementary Material

A Proofs of Propositions

Proof of Proposition 2.1:

Assume that $\bar{T}^f_{\Lambda,\Gamma}$ is the short-run technology $T^f_{\Lambda,\Gamma}$ without the variable input constraints $\mathbf{x}^v \geq \sum_{k=1}^K \delta z_k \mathbf{x}_k^v$. If $(\mathbf{x}^f, \mathbf{y}) \in T^f_{\Lambda,\Gamma}$, then it is clear that $(\mathbf{x}^f, \mathbf{y}) \in \bar{T}^f_{\Lambda,\Gamma}$. Moreover, if $(\bar{\mathbf{x}}^f, \bar{\mathbf{y}}) \in \bar{T}^f_{\Lambda,\Gamma}$, then there exist $\bar{\mathbf{z}} \in \Lambda, \bar{\delta} \in \Gamma$ such that $\bar{\mathbf{x}}^f \geq \sum_{k=1}^K \bar{\delta}\bar{z}_k \mathbf{x}_k^f, \bar{\mathbf{y}} \leq \sum_{k=1}^K \bar{\delta}\bar{z}_k \mathbf{y}_k$ and also we have $\sum_{k=1}^K \bar{\delta}\bar{z}_k \mathbf{x}_k^v = \bar{\mathbf{x}}^v$. Since $\mathbf{x}^v$ is a arbitrary variable without any upper bound in technology $T^f_{\Lambda,\Gamma}$, by considering $\mathbf{x}^v \geq \bar{\mathbf{x}}^v$ in technology $T^f_{\Lambda,\Gamma}$, we have $(\bar{\mathbf{x}}^f, \bar{\mathbf{y}}) \in T^f_{\Lambda,\Gamma}$. Therefore, we have $T^f_{\Lambda,\Gamma} = \bar{T}^f_{\Lambda,\Gamma}$. In a similar way, we have $P^f_{\Lambda,\Gamma}(\mathbf{x}_p^f) = \bar{P}^f_{\Lambda,\Gamma}(\mathbf{x}_p^f)$ where $\bar{P}^f_{\Lambda,\Gamma}(\mathbf{x}_p^f)$ is the short-run output set $P^f_{\Lambda,\Gamma}(\mathbf{x}_p^f)$ without the variable input constraints $\mathbf{x}^v \geq \sum_{k=1}^K \delta z_k \mathbf{x}_k^v$.

Proof of Proposition 2.2:

For a given fixed input $\mathbf{x}_p^f$ of DMU_p, the corresponding output $\mathbf{y}_p$ of DMU_p belongs to $P^f_{\Lambda,\Gamma}(\mathbf{x}_p^f)$, i.e., $\mathbf{y}_p \in P^f_{\Lambda,\Gamma}(\mathbf{x}_p^f)$, thus the short run output set $P^f_{\Lambda,\Gamma}(\mathbf{x}^f)$ is nonempty. Based on (6), it is clear that the short run output set $P^f_{\Lambda,\Gamma}(\mathbf{x}^f)$ is a closed set. Now, we show that $P^f_{\Lambda,\Gamma}(\mathbf{x}_p^f)$ is a bounded set. Let $P^f_{\Lambda,\Gamma}(\mathbf{x}_p^f)$ be an unbounded set, then for each $M > 0$, there exist $\mathbf{y}_M \in P^f_{\Lambda,VRS}(\mathbf{x}_p^f)$ such that $||\mathbf{y}_M|| > M$. Therefore, there exist $\delta > 0$ and $(z_1, ..., z_K) \in \Lambda$ such that $\mathbf{y}_M \leq \sum_{k=1}^K \delta z_k \mathbf{y}_k$ and since $||\mathbf{y}_M|| > M > 0$, we have $\sum_{k=1}^K \delta z_k \mathbf{y}_k \neq 0$. Thus, there exist $k' \in \{1, ...K\}$ such that $\delta z_{k'} > 0$. Based on (D.4) each producer uses some fixed input, it results $\mathbf{x}_{k'}^f \neq 0$. Therefore, $\delta z_{k'} \mathbf{x}_{k'}^f \neq 0$. If $M \to \infty$, then $||\mathbf{y}_M|| \to \infty$ and $\delta z_{k'} \to \infty$. Therefore, $\delta z_{k'} \mathbf{x}_{k'}^f \to \infty$. Since $\mathbf{x}_p^f \geq \sum_{k=1}^K \delta z_k \mathbf{x}_k^f$ and $\mathbf{x}_p^f$ is finite, we have a contradiction. Therefore, $P^f_{\Lambda,\Gamma}(\mathbf{x}_p^f)$ is a closed and bounded set.

Proof of Proposition 2.3:

Since $\delta \geq 0, z_k \geq 0$ for all $k \in \{1, ..., K\}$ and $\mathbf{y}_k \geq \mathbf{0}$, therefore, we always have $\sum_{k=1}^K \delta z_k \mathbf{y}_k \geq 0$ in optimality. Thus, the constraint $\sum_{k=1}^K \delta z_k \mathbf{y}_k \geq 0$ is redundant and can be removed from the short-run input set $L_{\Lambda,\Gamma}(\mathbf{0})$.

Proof of Proposition 2.4:

Based on axiom (T.3), for both VRS and CRS cases, we have

$$L_{\Lambda,\Gamma}(\mathbf{y}_{min}) \subseteq L_{\Lambda,\Gamma}(\mathbf{y}^\epsilon) \subseteq L_{\Lambda,\Gamma}(\mathbf{0}). \tag{A.1}$$

(i) Let $0 \leq \mathbf{y}^\epsilon \leq \mathbf{y}_{min}$. Assume that $\mathbf{x} \in L_{\Lambda,VRS}(\mathbf{y}^\epsilon)$. Therefore, there exist $\mathbf{z} = (z_1, ...z_k)$ such that $\mathbf{x} \geq \sum_{k=1}^K z_k \mathbf{x}_k$, $\mathbf{y}^\epsilon \leq \sum_{k=1}^K z_k \mathbf{y}_k$ and $\sum_{k=1}^K z_k = 1$. Since $\mathbf{y}_{min} = \min\limits_{k=1,...,K} \mathbf{y}_k$,

$\sum_{k=1}^{K} z_k\mathbf{y}_k \geq \sum_{k=1}^{K} z_k\mathbf{y}_{min} = \mathbf{y}_{min}$. Therefore, $\mathbf{x} \in L_{\Lambda,VRS}(\mathbf{y}_{min})$. Hence,

$$L_{\Lambda,VRS}(\mathbf{y}_{min}) \subseteq L_{\Lambda,VRS}(\mathbf{y}^{\epsilon}). \tag{A.2}$$

Based on (A.1) and (A.2), we have $L_{\Lambda,VRS}(\mathbf{y}_{min}) = L_{\Lambda,VRS}(\mathbf{y}^{\epsilon})$. In a similar way, we can prove that $L_{\Lambda,VRS}(\mathbf{y}^{\epsilon}) = L_{\Lambda,VRS}(\mathbf{0})$.

(ii) Based on (A.1), we have $L_{\Lambda,CRS}(\mathbf{y}^{\epsilon}) \subseteq L_{\Lambda,CRS}(\mathbf{0})$. Since $\mathbf{0} \in L_{\Lambda,CRS}(\mathbf{0})$ and $\mathbf{0} \notin L_{\Lambda,CRS}(\mathbf{y}^{\epsilon})$, hence $L_{\Lambda,CRS}(\mathbf{y}^{\epsilon}) \subset L_{\Lambda,CRS}(\mathbf{0})$.

Based on (A.1), we have $L_{\Lambda,CRS}(\mathbf{y}_{min}) \subseteq L_{\Lambda,CRS}(\mathbf{y}^{\epsilon})$. Assume that$(\mathbf{x}_p, \mathbf{y}_p)$ is an observed unit. Let $\theta^* = \min\{\theta \mid \theta\mathbf{x}_p \in L_{\Lambda,CRS}(\mathbf{y}_{min})\}$ and $\mathbf{x}^* = \theta^*\mathbf{x}_p$. Thus, $(\mathbf{x}^*, \mathbf{y}_{min}) \in T$.

Since $\mathbf{y}^{\epsilon} < \mathbf{y}_{min}$, there exist $\alpha < 1$ such that $\mathbf{y}^{\epsilon} < \alpha\mathbf{y}_{min}$. Thus, $\alpha\mathbf{x}^* \in L_{\Lambda,CRS}(\mathbf{y}^{\epsilon})$. If $L_{\Lambda,VRS}(\mathbf{y}_{min}) = L_{\Lambda,VRS}(\mathbf{y}^{\epsilon})$, then $\alpha\mathbf{x}^* \in L_{\Lambda,VRS}(\mathbf{y}_{min})$. Therefore, $\alpha\theta^*\mathbf{x}_p \in L_{\Lambda,VRS}(\mathbf{y}_{min})$ where $\alpha\theta^*\mathbf{x}_p < \theta^*\mathbf{x}_p$ which it is a contradiction with optimal value θ^*.

Proof of Proposition 2.5:

Based on axiom (T.3), for both VRS and CRS cases, we have

$$P_{\Lambda,\Gamma}^{\mathbf{x}_{max}} \subseteq P_{\Lambda,\Gamma}^{\mathbf{x}^{\epsilon}} \subseteq P_{\Lambda,\Gamma}. \tag{A.3}$$

(i) Assume that $\mathbf{y} \in P_{\Lambda,VRS}$. Therefore, there exist $\mathbf{z} = (z_1, ...z_k)$ and $\mathbf{x}$ such that $\mathbf{x} \geq \sum_{k=1}^{K} \mathbf{x}_k z_k$, $\mathbf{y} \leq \sum_{k=1}^{K} \mathbf{y}_k z_k$ and $\sum_{k=1}^{K} z_k = 1$. Since $\mathbf{x}_{max} = \max\limits_{k=1,...,K} \mathbf{x}_k$, we have $\sum_{k=1}^{K} \mathbf{x}_k z_k \leq \sum_{k=1}^{K} \mathbf{x}_{max} z_k = \mathbf{x}_{max} \leq \mathbf{x}^{\epsilon}$. Therefore, $\mathbf{y} \in P_{\Lambda,VRS}^{\mathbf{x}^{\epsilon}}$ and $\mathbf{y} \in P_{\Lambda,VRS}^{\mathbf{x}_{max}}$. Hence,

$$P_{\Lambda,VRS} = P_{\Lambda,VRS}^{\mathbf{x}_{max}} = P_{\Lambda,VRS}^{\mathbf{x}^{\epsilon}}. \tag{A.4}$$

Also, since in the VRS case we have $\sum_{k=1}^{K} z_k = 1$, we have $P_{\Lambda,VRS}^{\mathbf{x}^{\epsilon}} \subset \mathbb{R}_+^M$.

(ii) It is clear that we have $P_{\Lambda,CRS} = \mathbb{R}_+^M$ under the CRS case. Based on relation (A.3), we have $P_{\Lambda,CRS}^{\mathbf{x}_{max}} \subseteq P_{\Lambda,CRS}^{\mathbf{x}^{\epsilon}} \subseteq P_{\Lambda,CRS}$. We need to prove that $P_{\Lambda,CRS}^{\mathbf{x}^{\epsilon}} \subset P_{\Lambda,CRS} = \mathbb{R}_+^M$.

Therefore, we show that $P_{\Lambda,CRS}^{\mathbf{x}^{\epsilon}}$ is a bounded set. Let $P_{\Lambda,CRS}^{\mathbf{x}^{\epsilon}}$ be an unbounded set, then for each $M > 0$, there exist $\mathbf{y}_M \in P_{\Lambda,CRS}^{\mathbf{x}^{\epsilon}}$ such that $||\mathbf{y}_M|| > M$. Since $\mathbf{y}_M \in P_{\Lambda,CRS}^{\mathbf{x}^{\epsilon}}$, there exist $\delta > 0$ and $(z_1, ..., z_K) \in \Lambda$ such that $\mathbf{y}_M \leq \sum_{k=1}^{K} \delta z_k \mathbf{y}_k$. If $M \to \infty$, then $||\mathbf{y}_M|| \to \infty$. Hence, there exists $k' \in \{1, ...K\}$ such that $\delta z_{k'} \to \infty$. Based on (D.3) each producer utilises a positive amount of at least one input to produce a positive amount of at least one output, it leads to $\mathbf{x}_{k'} \neq 0$. Therefore, $\sum_{k=1}^{K} \delta z_k \mathbf{x}_k \to \infty$. Since $\mathbf{x}^{\epsilon} \geq \sum_{k=1}^{K} \delta z_k \mathbf{x}_k$ and $\mathbf{x}^{\epsilon}$ is finite, we have a contradiction. Therefore, $P_{\Lambda,CRS}^{\mathbf{x}^{\epsilon}}$ is a bounded set.

Proof of Proposition 2.6:

The result follows directly from parts (i) and (ii) of Proposition 2.5.

Proof of Proposition 2.7:

(i) Based on determining models (13) and (14), since $(\mathbf{x}_p^f, \mathbf{x}_p^v) \in L_{\Lambda,VRS}(\mathbf{y}_p)$ and $\mathbf{x}_p \in L_{\Lambda,VRS}(\mathbf{0})$, respectively. Therefore, there is a feasible solution with $\theta = 1$ for both determining models (13) and (14). Also, since both models aim to minimize θ, we have $DF_i(\mathbf{x}_p, \mathbf{0} \mid \Lambda, VRS) \leq 1$ and $DF_i^{SR}(\mathbf{x}_p^f, \mathbf{x}_p^v, \mathbf{0} \mid \Lambda, VRS) \leq 1$. Also, since in the VRS case we have $\sum_{k=1}^{K} z_k = 1$, we have $0 < DF_i(\mathbf{x}_p, \mathbf{0} \mid \Lambda, VRS)$ and $0 < DF_i^{SR}(\mathbf{x}_p^f, \mathbf{x}_p^v, \mathbf{0} \mid \Lambda, VRS)$.

(ii) Since $\mathbf{z} = (z_1, ..., z_K) = \mathbf{0}$ is a feasible solution of determining models (13) and (14) under the CRS case, we have $DF_i(\mathbf{x}_p, \mathbf{0} \mid \Lambda, CRS) = DF_i^{SR}(\mathbf{x}_p^f, \mathbf{x}_p^v, \mathbf{0} \mid \Lambda, CRS) = 0$.

Proof of Proposition 2.8:

The result follows directly from parts (i) and (ii) of Proposition 2.4.

Proof of Proposition 2.9:

The result follows directly from parts (i) and (ii) of Proposition 2.4.

Proof of Proposition 4.1:

The feasibility of model (30) follows directly from the feasibility of determining model $DF_o^f(\mathbf{x}_p^f, \mathbf{y}_p \mid \Lambda, \Gamma)$. If $U^I \leq 1$, then we have $\sum_{p=1}^{K} \mathbf{x}_p^v \leq \theta \sum_{p=1}^{K} \bar{\mathbf{x}}_p^v \leq \sum_{p=1}^{K} \bar{\mathbf{x}}_p^v$. Thus, the optimal value of model (30) is a feasible solution of the system of equations (29).
Now, to prove the inverse part of this proof, assume that the system of equations (29) is feasible and $(z_k^{p*}, \mathbf{x}_p^{v*})$ is a feasible solution of the system of equations (29). Therefore, $(\theta^* = 1, z_k^{p*}, \mathbf{x}_p^{v*})$ is a feasible solution of model (30) with objective value $\theta* = 1$. Since the objective function of model (30) is minimising, its optimum value is smaller or equal to unity (i.e., $U^I \leq 1$).

Proof of Proposition 4.2:

Assume that model (32) is feasible with optimal solution $(\theta_p^*, z_k^{p*}, \mathbf{x}_p^{v*})$. If $\bar{\lambda} < L^I$, then

$$\sum_{p=1}^{K} \mathbf{x}_p^{v*} \leq \bar{\lambda} \sum_{p=1}^{K} \bar{\mathbf{x}}_p^v < L^I \sum_{p=1}^{K} \bar{\mathbf{x}}_p^v.$$

Therefore, $(\hat{\theta} = \bar{\lambda}, \hat{z}_k^p = z_k^{p*}, \hat{\mathbf{x}}_p^v = \mathbf{x}_p^{v*})$ is a feasible solution of model (34) with objective value $\hat{\theta} = \bar{\lambda} < L^I$ which is a contradiction.
Now, to prove the inverse part of this proof, assume that $\bar{\lambda} \geq L^I$ and $(\theta_p^* = L^I, z_k^{p*}, \mathbf{x}_p^{v*})$ is a feasible solution of model (34). Therefore, $\sum_{p=1}^{K} \mathbf{x}_p^v \leq L^I \sum_{p=1}^{K} \bar{\mathbf{x}}_p^v \leq \bar{\lambda} \sum_{p=1}^{K} \bar{\mathbf{x}}_p^v$. Thus, $(\theta_p^* = 1, z_k^{p*}, \mathbf{x}_p^{v*})$ is a feasible solution of model (32) and this completes the proof.

Proof of Proposition 4.3:

The feasibility of model (35) follows directly from the feasibility of determining model $DF_i^{SR}(\mathbf{x}_p^f, \mathbf{x}_p^v, \mathbf{y}_p^\epsilon \mid \Lambda, \Gamma)$.

Proof of Proposition 5.1:

(i) By taking the limit for $\mathbf{y}^\epsilon \to \mathbf{0}$, we have $PCU_i^{SR}(\mathbf{x}_p, \mathbf{x}_p^f, \mathbf{y}_p, \mathbf{y}^\epsilon \mid \Lambda, CRS) = PCU_i^{SR}(\mathbf{x}_p, \mathbf{x}_p^f, \mathbf{y}_p \mid \Lambda, CRS)$. Hence, the result follows directly by combining (21) together with Proposition 2.7(ii).

(ii) By taking the limit for $\mathbf{y}^\epsilon \to \mathbf{0}$, we have $PCU_i^{LR}(\mathbf{x}_p, \mathbf{y}_p, \mathbf{y}^\epsilon \mid \Lambda, CRS) = PCU_i^{LR}(\mathbf{x}_p, \mathbf{y}_p \mid \Lambda, CRS)$. Hence, the result follows directly by combining (3.8) together with Proposition 2.7(ii).

Proof of Proposition 5.2:

By taking the limit for $\mathbf{y}^\epsilon \to \mathbf{0}$, we have $PCU_o^{LR}(\mathbf{x}_p, \mathbf{y}_p, \mathbf{x}^\epsilon \mid \Lambda, CRS) = PCU_o^{LR}(\mathbf{x}_p, \mathbf{y}_p, \mid \Lambda, CRS)$. Hence, the result follows directly by combining (22) together with Remark 2.2.

Proof of Proposition 5.3:

(i) By taking the limit for $\bar{\lambda} \longrightarrow \infty$, we have $ADF_o(\mathbf{x}_p, \mathbf{y}_p, \bar{\lambda} \mid \Lambda, CRS) \longrightarrow DF_o(\mathbf{y}_p \mid P_{\Lambda,CRS})$ where $DF_o(\mathbf{y}_p \mid P_{\Lambda,CRS}) = \infty$. Therefore, we have

$$\begin{aligned} \lim_{\bar{\lambda}\to\infty} APCU_o^{LR}(\mathbf{x}_p, \mathbf{y}_p, \bar{\lambda} \mid \Lambda, CRS) &= PCU_o^{LR}(\mathbf{x}_p, \mathbf{y}_p \mid \Lambda, CRS) \\ &= \tfrac{DF_o(\mathbf{x}_p, \mathbf{y}_p \mid \Lambda, CRS)}{DF_o(\mathbf{y}_p \mid P_{\Lambda,CRS})} = 0. \end{aligned} \tag{A.5}$$

(ii) By taking the limit for $\bar{\lambda} \longrightarrow \infty$, we have $ADF_o(\mathbf{x}_p, \mathbf{y}_p, \bar{\lambda} \mid \Lambda, VRS) \longrightarrow DF_o(\mathbf{y}_p \mid P_{\Lambda,VRS})$. Therefore, we have

$$\lim_{\bar{\lambda}\to\infty} APCU_o^{LR}(\mathbf{x}_p, \mathbf{y}_p, \bar{\lambda} \mid \Lambda, VRS) = \frac{DF_o(\mathbf{x}_p, \mathbf{y}_p \mid \Lambda, VRS)}{DF_o(\mathbf{y}_p \mid P_{\Lambda,VRS})} = PCU_o^{LR}(\mathbf{x}_p, \mathbf{y}_p \mid \Lambda, VRS).$$