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Endogenous Learning in Multi-Sector Economies

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Endogenous Learning in Multi-Sector Economies

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Abstract

Consider a multi-sector general equilibrium model where firms have incomplete information about the returns to scale of their production and where that information is sequentially updated once real production is observed. What is the impact of these learning dynamics on the market-wise equilibrium objects? Under which conditions are firms able to efficiently learn their actual returns to scale? At which rate does this learning happen? In this work, we analyze endogenous learning mechanisms and their implications for the market-wise equilibrium objects in the multi-sector model. Our results shed light on how idiosyncratic shocks translate into the learning dynamics of firms returns to scale. Particularly, we uncover the advantages and disadvantages of the maximum a-posteriori estimation as a learning approach and we observe that all the relevant information in the learning dynamics is encoded in the input decisions and the manner in which input decisions are taken. We deduce conditions under which firms are able to learn the actual returns to scale. Using the notion of centrality in the multi-sector network, we uncover a price mechanism which is consistent not only with the correct knowledge of the returns to scale, but also with any converging sequence of belief on the returns to scale. On the empirical side, the proposed analysis of the endogenous learning dynamics is complemented with a statistical approach that allows testing the presence and level of learning using available input-output data. The empirical figures reveal the presence of sizable learning processes (driven by underestimations and overestimations of the returns to scale parameters) in different sectors.

Keywords: Mathematical Economics, Multi-sector general equilibrium model, Incomplete information, Returns to scale, Maximum a-posteriori estimation

JEL codes: D5, D51, D83, C11, C13

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1 Introduction

In economics, learning is designated as the inference agents draw from observing the appropriateness of their actions (or those of others) to the external environment (Dasgupta and Stiglitz, 1988; Kaelbling et al., 1996; Penczynski, 2017; Mossel et al., 2020). In fact, consumers and firms usually possess incomplete knowledge of some of their payoff-relevant parameters, yet they are sometimes able to observe the outcomes of their actions and learn from it.

As noted more than half-a-century ago by Arrow (1962), "the role of experience in increasing productivity has not gone unobserved, though the relation is yet to be absorbed into the main corpus of economic theory." To illustrate how much the economic impact of knowledge acquisition was widespread and allegedly on the rise, Arrow (1962) showcased manufacturing processes in the aeronautical engineering industry, where the amount of labor-hour required to assemble an airplane body appeared to decrease as the number of units previously assembled increased.

While it is by now incontrovertible that learning can explain many economic phenomena, such as herding (Banerjee, 1992; Bikhchandani et al., 1992; Smith and Sørensen, 2000), financial turmoil (Scharfstein and Stein, 1990; Welch, 1992; Collin-Dufresne et al., 2016), and diffusion of innovation (Duan et al., 2009), this concept has not been extensively embedded in the analysis of general equilibrium.

To address this problem by means of an integrated modelling framework, we present a dynamic multi-sector model in which a collection of representative firms operates in a noisy environment and is unaware of the values of certain profit-relevant parameters. Specifically, we consider the uncertainty in the returns to scale and embed the dynamic multi-sector model into the Bayesian framework for optimal information collection and processing (El-Gamal and Sundaram, 1993; Cogley and Sargent, 2008). The proposed adaptation mechanism relies on an incomplete but perfect information setting, where firms dynamically observe the realized production outcomes after taking input decisions, while lacking a complete knowledge of their own returns to scale.

A long line of contributions has addressed the estimation of the returns to scale (Golany and Yu, 1997; Basu and Fernald, 1997; Banker et al., 2004; Ackerberg et al., 2015), not only for its important relationships with externalities and public goods (Starrett, 1977), but also for their impact on input missallocation (Gong and Hu, 2016). The incorrect knowledge of its own returns to scale can in fact induce a company to miss the optimal selection of production inputs, with major consequences at the macro level (Baqaee and Farhi, 2020).

Our quest for firms' ability to discover their own returns to scale from the dynamic observation of realized noisy productions is thus driven by the auspicious conjecture that this form of input missallocation can be asympthotically circumvented by a learning mechanism when partial information is dynamically revealed.

To detail this mechanism, our model assumes that in each production period, firms (i) make input decisions based on their current beliefs (point-wise estimation) of the returns to scale, (ii) observe their realized production, and (iii) compute the Bayesian posterior of their returns to scale to be used in the next period.

Hence, this approach mirrors a reinforcement learning procedure where the mismatch between a firm's observed production and the one expected for different returns to scale induces a performance metric (likelihood function) quantifying the correctness of a firm's belief at each period. This is in line with the recent work of Beggs (2022), in which agents evaluate outcomes relative to reference points (point-wise estimation), as well as with a long stream of contributions assessing the statistical properties of the maximizer of the Bayesian posterior distribution (Robert et al., 2007; Bolstad and Curran, 2016; Bassett and Deride, 2019). This maximum a-posteriori (hereafter referred to as MAP) estimation constitutes the main distinction between a fully Bayesian procedure and our proposed reinforcement learning procedure.

To outline the main insights of this analysis, our results shed light on three aspects. Firstly, all the relevant information in the learning dynamics (i.e., the information required to produce an update in firms' belief) is encoded in the input decisions. In other words, conditioned on input decisions, the learning path of the belief sequence is invariant with respect to labour, prices and household consumption. Secondly, we explore the advantages and disadvantages of employing the MAP estimator. On one hand, we derive conditions in closed-form that ensure firms can accurately learn the true returns to scale (referred to as the belief sequence converging to the actual returns to scale). These conditions are primarily linked to the approach used for making input decisions. On the other hand, we also demonstrate that the probability that the MAP estimator is not defined increases with the magnitude of idiosyncratic productivity shocks and decreases with the uncertainty in of the prior distribution. Finally, the mismatch between the true (unknown) returns to scale and those estimated by firms has a critical effect on the expected firms revenues. This, in turn, plays a crucial role in input misallocations. In this vein, using the notion of centrality in the multi-sector network, we uncover a price mechanism which is consistent with any converging sequence of belief on the returns to scale. This price mechanism supports the correct firms prediction of their own expected revenue, which allows mitigating input misallocations.

On the empirical side, the deduced market-wise equilibrium objects and the characterization of the learning dynamics have direct bearing on the construction of a least square estimation approach, that allows calibrating the parameters of the multi-sector model based on real data. For this purpose, we use the input-output accounts spanning the 1999-2019 period from the Bureau of Economic Analysis, and estimate industry level divergences between the belief and the true returns to scale. This reveals the presence of sizable learning processes in different sectors, that are driven by underestimations and overestimations of the returns to scale parameters.

The rest of this paper is organized as follows. Section 2 describes some further connections with the literature. Section 3 introduces the proposed modelling approach for a dynamic multi-sector input-output economy and its equilibrium conditions. Section 4 establishes our main results on the incomplete information and learning dynamics. Section 5 analyses the implications of the learning dynamics on prices and material input missallocation. Section 6 provides an empirical grounding for our results, based on a least square estimation approach and on real data from the Bureau of Economic Analysis. Section 7 concludes the paper. All the mathematical proofs of the propositions are reported in Appendix A.

2 Related literature

This work is connected to two streams of literature summarized in this section, highlighting the relationship to our contribution.

Input-output models. It is reasonable to believe that for decades the macroeconomic tendency to aggregate idiosyncratic variations stood in the way of multi-sector dynamic general equilibrium models with endogenous learning (Leontief, 1970). In point of fact, in an economy consisting of n sectors, the diversification argument by Lucas (1977) has been frequently used to abstract away from the amazingly sophisticated yet harmonious network of interconnected agents. This line of reasoning prevented macro-economists exploring the granularity of inter-sectoral networks under a statistical argument that micro-economic shocks average out across sectors when n grows.[1]

Motivated by empirical findings using micro-level data, an increasing emphasis on firm-level decisions has emerged in the last few decades (Coulson and Rushen, 1995). In this context, two modelling designs departing from the diversification argument are worth mentioning. Firstly, Horvath (2000) presented a multi-sector dynamic general equilibrium model of business cycles, where "trade among sectors provides a strong synchronization mechanism for shocks due to the limited, but locally intense, interaction that is characteristic of such input trade flows". He proved that this limited interaction leads to a postponement of the law of large numbers in the variance of aggregate production. Secondly, Gabaix (2011) showed that the diversification argument breaks down if the distribution of firm sizes is fat-tailed, so that idiosyncratic sector-level shocks can explain an important part of aggregate movements. Particularly, when firm size is power-law distributed, the conditions under which one derives the central limit theorem break down.

On the one hand, both approaches challenge the diversification argument, by pointing out economically relevant conditions under which it cannot be applied. On the other hand, they support the need of zooming into the complex granularity of input-output economic structures, to explore the non-negligible micro-founded effects from which learning originates.

Building on the work of Horvath (2000), a static multi-sector general equilibrium model has been proposed by Acemoglu et al. (2012), who demonstrated that "the interplay of idiosyncratic microeconomic shocks and sectoral heterogeneity results in systematic departures in the likelihood of large economic downturns relative to what is implied by the normal distribution".[2]

This approach has the potential to be further generalized. For instance, a time dimension has been included in a follow-up paper by Acemoglu et al. (2017), building on the real business cycle analysis by Long Jr and Plosser (1983), in which it takes each firm one period to transform inputs to output. Due to its relevance for the scope of our work, this dynamic extension of the multi-sector general equilibrium model represents a point of reference for the study of a firm's endogenous learning along the analyzed time horizon. From this viewpoint, our work opens a new line of generalization for the modelling framework of Long Jr and Plosser (1983), Horvath (2000),

[1] In an economy hit by independent shocks at its n sectors, aggregate fluctuations would have a magnitude proportional to $1/\sqrt{n}$, a negligible effect at high levels of disaggregation.

[2] The authors showed that under certain supply-chain configurations, such as the star network, the law of large numbers fails and the aggregate output does not concentrate around a constant value.

and Acemoglu et al. (2012), extending this stream of literature towards the analyses of endogenous learning.

Learning-by-doing and social learning. Since the seminal work of Arrow (1962), different modelling approaches have been proposed to integrate the acquisition of knowledge in the main corpus of economic theory. For instance, Lucas (1988) has studied how increasing returns to embodied human capital can be explained based on endogenous learning mechanisms. Dasgupta and Stiglitz (1988) studied oligopolistic industries in the presence of learning-by-doing. D'Albis et al. (2012) study the impact of learning-by-doing on business cycle fluctuations.

As noted by Vives (1996), the initial goal of the learning literature in the economic context has been to emphasize that inefficient outcomes can happen even when consumers and firms act rationally. For example, agents may herd on a wrong action disregarding private information (Banerjee, 1992; Bikhchandani et al., 1992). This literature emphasizes market failures and feeds into a tradition of study of excess volatility and crashes in financial markets. Vives (1996) explored some social learning models and noticed that in all of them "information about the choices of other agents is obtained via noisy aggregates, prices or quantities". He analyzed the extent to which the learning dynamics is capable of casting a doubt on the reliance of the market mechanism.

In the last two decades, the economic literature involving the incomplete information and the acquisition of knowledge has progressively opened toward learning games and the comparative analysis of different learning mechanisms, especially focusing on the Bayesian versus non-Bayesian approaches (Jadbabaie et al., 2012; Molavi et al., 2018; Brandt et al., 2021). In this context, Mossel et al. (2020) propose a new equilibrium notion for learning games, under the aforementioned idea that economic agents make decisions on possible alternatives without full awareness of the outcomes of their actions. Therefore, they use their and others past experiences to learn. The query about the ability of economic agents to learn from observing the outcomes of their actions has been introduced in the economic literature in multiple stages by Easley and Kiefer (1988), El-Gamal and Sundaram (1993), and Wieland (2000), looking at different sides of the problem.

Despite this growing interest, the theoretical literature focusing on the implication of learning on the general equilibrium remains disconnected and sparse. In this respect, our contribution relates to the market-wise equilibrium implications of firms' ability to acquire knowledge, building on the idea that input decisions yield information on their appropriateness to the external environment (and, in turn, to the partially unknown production processes).

3 The model

3.1 Baseline definitions and notation

We introduce a dynamic multi-sector economy, closely resembling the modelling frameworks of Horvath (2000), Gabaix (2011), Acemoglu et al. (2012), and Acemoglu et al. (2017). This consists of a collection $\mathcal{N}$ of distinct sectors (with $|\mathcal{N}| = n$), each producing different goods over a finite time horizon of length T. The output of each sector is controlled by a representative firm that operates so as to maximize its expected profit at the end of the production periods.[3] Production does not take place instantaneously, but with a period delay following the purchase of inputs.[4]

In addition to the sectors, the economy comprises a representative consumer, taking decisions about a preferred consumption plan over the n goods, with the aim of maximizing its lifetime utility. We let $x_i(t)$, $c_i(t)$, and $p_i(t)$ be the corresponding production, consumption, and price of the i-th good at the end of the t-th production period, for $t = 1, \ldots, T$. In vector form, $\mathbf{x}(t) = [x_1(t) \ldots x_n(t)]^\top$, $\mathbf{c}(t) = [c_1(t) \ldots c_n(t)]^\top$ and $\mathbf{p}(t) = [p_1(t) \ldots p_n(t)]^\top$. We also define $\boldsymbol{l}(t) = [l_1(t) \ldots l_n(t)]^\top$ as the vector of labor quantities, measured on a time scale.

Concerning the matrix notation, we denote as $[A]_i$ the i-th row of a matrix A. Similarly, given a sequence of t matrices $\mathcal{A}(t) = \{A(\ell)\}_{\ell=1}^t$, we denote as $\mathcal{A}_i(t) = \{[A(\ell)]_i\}_{\ell=1}^t$ the sequence of their i-th row vectors.

3.2 The representative household and firms

The consumer problem is designed as a time-discounted logarithmic utility maximization, subject to the inter-temporal budget constraint:

$$\begin{aligned} &\max_{\{\mathbf{c}(t)\}_{t=1}^T} && \sum_{t=1}^{T} \rho^t \sum_{i \in \mathcal{N}} \kappa_i \log c_i(t) \\ &\text{subj. to} && \sum_{i \in \mathcal{N}} p_i(t) c_i(t) + \bar{p} a(t+1) \leq w(t) + (\bar{p} + \vartheta(t)) a(t), \quad \text{for } t = 1 \ldots T-1, \end{aligned} \tag{1}$$

where, for the sake of market completeness, we include a credit sector with interest rate $\bar{p} + \vartheta(t)$. Here $a(t)$ is the household's holding of the credit asset, $w(t)$ the salary corresponding to the labor income, $(\bar{p}+\vartheta(t))a(t)$ the available savings, $\vartheta(t)$ captures a dividend paid on endogenous investment at the t-th period. The exogenous constant $\rho \in (0, 1]$ denotes the households' discount factor, and κ_i the utility elasticity of product i, with $\sum_{i=1}^n \kappa_i = 1$. We assume the boundary conditions $a(T) = 0$, $w(1) = 0$ and $a(1)$ to be fixed exogenously (with $a(1) > 0$).

Each good in the economy can be either consumed or used in the next period by other sectors as an input for production. As in Horvath (2000) and Acemoglu et al. (2012, 2017), the sectors use Cobb-Douglas technologies. Thus, the output is

$$x_i(t) = \eta_i(t) \mu_i(t), \qquad \text{with} \qquad \mu_i(t) = (l_i(t))^{(1-\phi)} \left(\prod_{j=1}^{n} (y_{i,j}(t))^{\phi \alpha_{i,j}} \right)^{v_i}, \tag{2}$$

where $y_{i,j}(t)$ denotes the amount of production of sector j used as an input in sector i. The exogenous parameter $\phi \in [0, 1]$ establishes the material versus labor intensity in production. The inter-sectoral

[3] The simplifying notion of a representative firm is often used by macroeconomic models (Hartley and Hartley, 2002). By this is meant a hypothetical firm whose production is equal to the aggregate production of the sector as a whole, and whose inputs are equal to the aggregate inputs of the sector as a whole. At the empirical level, industry taxonomies (such as the Global Industry Classification Standard, or the Standard Industrial Classification) can be used to group companies, based on similar production processes and products.

[4] As stated by Sargan (1955), "[...] a period which measures the effect of the rate of interest on relative prices of inputs and outputs should be called the period of production. Alternatively, this period can be defined as the difference between an output period and an input period, each with respect to the plans of a marginal entrepreneur about to start a new firm."

input-output elasticities are summarized by $\phi\alpha_{i,j}v_i$, measuring the output response to a change in levels of production inputs, where $\alpha_{i,j}$ is the input-specific factor of this elasticity (with $\sum_{j=1}^n \alpha_{i,j} = 1$), and v_i quantifies the returns to scale of the material inputs in sector i. As studied in Section 4, the returns to scale value represents the unknown payoff-relevant parameter firms wish to learn by observing the outcomes (realized production) of their actions (input decisions).[5] We introduce the notation $\delta_{i,j} = \alpha_{i,j}v_i$ and let A be a $n\times n$ non-singular matrix with components $\alpha_{i,j}$, and Δ be a $n\times n$ matrix with components $\delta_{i,j}$. Similarly, $Y(t)$ is used to denote a $n \times n$ matrix with components $y_{i,j}(t)$. For each $i \in \mathcal{N}$, the random factors $\eta_i(t) \sim \log N(m_i, \sigma_i)$ are independent idiosyncratic productivity shocks, with $\log N(m_i, \sigma_i)$ denoting a log-Gaussian distribution with parameters m_i and σ_i. The scalar quantity m_i relates to the i-th firm productivity constant, whose expectation is $q_i = \mathbb{E}_{\boldsymbol{\eta}}[\eta_i(t)] = \exp(m_i + \sigma_i^2/2)$.

Consider the total expenditure on inputs (including the expenditure on labor) that the i-th representative firm has to pay at the t-th period $e_i(t)$, and its corresponding revenue $r_i(t)$ after carrying out the production process from t to $t+1$:

$$\begin{cases} e_i(t) &= w(t+1)l_i(t) + \displaystyle\sum_{j=1}^n p_j(t)y_{i,j}(t), \qquad \text{for } \ i \in \mathcal{N}, \quad t = 1 \dots T-1, \\ r_i(t) &= p_i(t+1)x_i(t), \qquad \text{for } \ i \in \mathcal{N}, \quad t = 1 \dots T-1, \end{cases} \tag{3}$$

where salaries are paid posterior to the realization of the production period. Building on (2) and (3), the firms' problem at the t-th period is to maximize the individual expected profits, by deciding the share of labor and material inputs to include in the production:

$$\max_{l_i(t), y_{i,1}(t), \dots, y_{i,n}(t)} \quad \mathbb{E}_{\boldsymbol{\eta}}\Big[(r_i(t) - e_i(t)) \ \Big| \ v_i \Big] \qquad \text{for } \ i \in \mathcal{N}, t = 1 \dots T-1, \tag{4}$$

where we make explicit the dependency with respect to the knowledge of v_i by firms. We also introduce the notation $\hat{r}_i(t) = \mathbb{E}_{\eta_i(t)}[r_i(t)] = p_i(t+1)q_i\mu_i(t)$, referring to the expected revenue. Production, consumption and material inputs are linked by the market clearing conditions:

$$\begin{cases} x_j(t) &= c_j(t+1) + \sum_{i\in\mathcal{N}} y_{i,j}(t+1), \qquad \text{for } j \in \mathcal{N}, \ t = 1 \dots T-2, \\ x_j(T-1) &= c_j(T), \qquad \text{for } j \in \mathcal{N}, \\ x_j(T) &= 0, \qquad \text{for } j \in \mathcal{N}. \end{cases} \tag{5}$$

Similarly, in accordance with Acemoglu et al. (2012, 2017), the total supply of labor is fixed and the representative household is endowed with one unit of labor, which is supplied inelastically:

$$\sum_{i\in\mathcal{N}} l_i(t) = 1, \qquad \text{for } t = 1 \dots T. \tag{6}$$

Firms and household are assumed to be price-takers, with prices determined by enforcing the clearing conditions (5). Dividends are determined by enforcing the period by period equality between firms total profit and household saving.

[5] The multi-sector general equilibrium models of Acemoglu et al. (2012) and Acemoglu et al. (2017) rely upon constant returns to scale (i.e., $v_i \sum_j \alpha_{i,j} = 1$), whereas this assumption is not set in our modelling design.

3.3 Market players' decisions

The learning mechanisms studied in this paper rely on the characterization of market players' decisions, and in particular material input decisions $Y(t)$, for $t = 1, \dots, T-1$. While our theoretical analysis of the learning dynamics is general (in the sense that its theoretical properties and implications are valid for any characterization of $Y(t)$) and can still be adopted when market players take sub-optimal decisions (based on heuristic algorithms or rules of thumb (Blonski, 1999)), this section characterizes market players' decisions by the exact utility maximization (problem (1)) and profit maximization (problem (4)).

Lemma 1 (Household's consumption)**.** *The utility maximizing household's consumption is:*

$$c_i(t)p_i(t) \;=\; \kappa_i \overline{w}(t), \qquad \text{for } i \in \mathcal{N}, \quad t = 1 \dots T-1, \tag{7}$$

where $\overline{w}(t) = w(t) - \bar{p}(a(t+1) - a(t)) + \vartheta(t)a(t)$. The optimal holding of the credit stock is achieved by satisfying the inter-temporal condition

$$\frac{\bar{p}}{\overline{w}(t)} = \frac{\rho(\bar{p} + \vartheta(t))}{\overline{w}(t+1)}, \qquad \text{for } 1 \le t \le T-2, \tag{8}$$

when wage $w(t+1)$ and dividend $\vartheta(t+1)$ are known by the household at time t, and

$$\frac{\bar{p}}{\overline{w}(t)} = \mathbb{E}\left[\frac{\rho(\bar{p} + \vartheta(t))}{\overline{w}(t+1)}\right], \qquad \text{for } 1 \le t \le T-2, \tag{9}$$

when wage $w(t+1)$ and dividend $\vartheta(t+1)$ are unknown by the household at time t.

From the second order recurrence relations (8) and(9), the equilibrium path of saving is determined given the exogenous conditions $a(1)$ and $a(T)$.

Lemma 2 (Firms' inputs)**.** *Let $\mathcal{N}_+ = \{i \in \mathcal{N} \,:\, v_i > 0\}$ and $\mathcal{N}_0 = \mathcal{N} \setminus \mathcal{N}_+ = \{i \in \mathcal{N} \,:\, v_i = 0\}$.*[6] *The demand for labor and material inputs are:*

$$l_i(t) = \frac{(1-\phi)}{\phi w(t+1)} q_i p_i(t+1)\mu_i(t), \quad \text{for } t = 1, \dots, T-2, \tag{10}$$

$$y_{i,j}(t) = \frac{v_i \alpha_{i,j}}{p_j(t)} q_i p_i(t+1)\mu_i(t), \qquad \text{for }, \; t = 1, \dots, T-2, \tag{11}$$

for each $i \in \mathcal{N}_+$ and

$$l_i(t) = \left(\frac{(1-\phi)p_i(t+1)}{w(t+1)}\right)^{\frac{1}{\phi}}, \qquad \text{for } t = 1, \dots, T-2, \tag{12}$$

$$y_{i,j}(t) = 0, \qquad \text{for } \; t = 1, \dots, T-2, \tag{13}$$

for each $i \in \mathcal{N}_0$.

Lemma 2 illustrates the functional dependency between input decisions and returns to scale, constituting the fundamental building block of the learning dynamics studied in Section 4. A special attention is given to the discontinuity at point $v_i = 0$, having major consequences on firms

[6] As discussed in Section 4, the case $v_i = 0$ is also a possible realization of the learning dynamics.

ability to learn. Building on Lemmas 1 and 2, a characterization of equilibrium prices and salaries consistent with Horvath (2000) is provided in Section 5, analyzing the implications of the learning dynamics on market-wise equilibrium quantities.[7]

4 Uncertainty over payoff-relevant parameters and learning

We now introduce firms' uncertainty about the profit-relevant parameters $\{v_i\}_{i\in\mathcal{N}}$, and study learning mechanisms for their dynamic estimation from the observed production.

To do so, we embed the proposed multi-sector general equilibrium model into the framework of information collection and processing introduced by El-Gamal and Sundaram (1993) (and further explored by Cogley and Sargent (2008)). They presented the optimization problem facing a single infinitely-living Bayesian agent, who alternatingly combines a decision stage and a learning stage. More precisely, the agent is unaware of the values of some payoff-relevant parameters and wishes to maximize the expected discounted lifetime reward. Its actions yield information on these unknown parameters through the observed rewards (El-Gamal and Sundaram, 1993).

Adapting this framework to our multi-sector model, we let $v_i^{(*)}$ denote the true value of the returns to scale parameter and $v_i(t)$ its corresponding firm's estimation (for each $i \in \mathcal{N}$) at the t-th period. Consequently, we update the notation $\delta_{i,j}$ and Δ to become $\delta_{i,j}(t) = v_i(t)\alpha_{i,j}$ and $\Delta(t)$, respectively. Similarly, $\delta_{i,j}^{(*)} = v_i^{(*)}\alpha_{i,j}$ and $\Delta^{(*)}$ are defined in an analogous way. Next, to lighten the mathematical presentation, the following notation is used throughout this section:

$$\varepsilon_i(t) = \log(\eta_i(t)), \qquad z_i(t) = \log\left(\prod_{j=1}^{n}(y_{i,j}(t))^{\phi\alpha_{i,j}}\right) \qquad \text{and} \qquad s_i(\ell) = \varepsilon_i(\ell) + v_i^{(*)} z_i(\ell). \tag{14}$$

Therein, firms take decisions based on $v_i(t)$, while assessing the appropriateness of their beliefs to the realized production when constructing a probability distribution for $v_i(t+1)$ (the next period belief of firm i). This is done by invoking the Bayesian rule and a collection of learning assumptions.

Assumption 1. *At each production period, firms use the maximum a-posteriori as a point-wise estimator of the unknown* $v_i^{(*)}$.

The adoption of a point-wise estimation method when alternatingly combining a decision stage and a learning stage, relies on the anticipated utility approach of Kreps (1998) and Cogley and Sargent (2008). In this approach, firms treat the profit-relevant parameter $v_i(t)$ as a random quantity when they learn but as a constant when they take decisions. Conversely, a full Bayesian procedure would regard it as a random quantity both for learning and for decision-making.[8]

We focus on two specifications of the proposed learning method: a long-memory approach, where firms keep track of the whole history of past productions when building the probability distribution of $v_i(t)$ (as studied in Subsection 4.1); a short-memory approach, where firms only record one

[7] Bargaining formulations to simultaneously determine input decisions and prices in non-competitive markets can be considered as an extension (Benita et al., 2022).

[8] As noted by Cogley and Sargent (2008), a full Bayesian procedure is mathematically intractable for most economic problems, so that they studied the goodness of the anticipated utility for different economic models.

previous period when building $v_i(t)$ (as studied in Subsection 4.2). As explicitly demonstrated in these subsections, since the proposed multi-sector model does not include durable capital inputs, the learning dynamics (which ultimately reflects input decisions) does not depend on weather the time horizon is finite or infinite.

4.1 Long-memory learning

A long-memory method is a point-wise estimation approach constructed by enforcing the following assumptions on the information set and the initial knowledge.

Assumption 2. *At period t, the firm's i information set is given by the series of past realized productions up to the current period: $\mathcal{I}_i(t-1) = \{x_i(t-1), x_i(t-2), \ldots, x_i(1)\}$, for $2 \le t \le T$.*

Assumption 3. *Firms have an initial knowledge $\pi_{0,i}(v_i)$ quantified as a zero-truncated Gaussian with parameters $v_i^{(0)}$ and τ_i.*[9]

With respect to Assumption 2, the realized production encodes all required information (in the sense of statistical sufficiency) for the likelihood characterization. In fact, the likelihood function of the returns to scale parameter $v_i(t)$ is the probability distribution over the realized productions, which is induced by the idiosyncratic productivity shocks $\eta_i(1), \ldots, \eta_i(t-1)$:

$$\mathscr{L}(v_i(t); \mathcal{I}_i(t-1)) = \prod_{h=1}^{t-1} \frac{1}{(2\pi\sigma_i)^{\frac{1}{2}}} \exp\left(-\frac{1}{2\sigma_i^2}\left(\log x_i(h) - \log \mu_i(h)\right)^2\right), \qquad \text{for } 2 \le t \le T, \tag{15}$$

where $\mu_i(t)$ has been defined in (2) and represents the deterministic part of the Cobb-Douglas production. Hence, $\mathscr{L}(v_i(t); \mathcal{I}_i(t))$ quantifies the mismatch between the log-production of firm i observed up to period t, and the one it would expect for different values of $v_i(t)$. This can be seen as a way of assessing the goodness of firm i's knowledge of its production function, or as an environment reward to the appropriateness of its input allocation.[10]

The Bayesian posterior distribution of $v_i(t)$ at the t-th period is

$$\pi_{t,i}(v_i(t)) \;=\; \begin{cases} \dfrac{\mathscr{L}(v_i(t); \mathcal{I}_i(t-1))\pi_{0,i}(v_i(t))}{\int \mathscr{L}(v_i(t); \mathcal{I}_i(t-1))\pi_{0,i}(v_i(t))\; dv_i(t)} & \text{if } t > 1, \\ \pi_{0,i}(v_i(t)) & \text{if } t = 1. \end{cases} \tag{16}$$

[9] It is worth noting that $v_i(t)$ represents a MAP estimator, while $\pi_{t,i}(v_i)$ denotes its probability density function.

[10] In the context of the empirical production literature, the likelihood function $\mathscr{L}(v_i(t); \mathcal{I}_i(t))$ has been used since the seminal paper of Marschak and Andrews (1944) to estimate the returns to scale from realized production. Contextually, the described learning approach compounds a reinforcement learning mechanism, defined upon (i) a state space of the system, (ii) a set of actions to be taken by firms, (iii) a policy, and (iv) a performance metric.

(i) The state space of the system is the set of feasible returns to scale parameters.

(ii) The actions taken by firm i at period t are $l_i(t)$ and $y_{i,1}(t), \ldots, y_{i,n}(t)$.

(iii) The policy defining the decision criteria at each state visited by the system is the profit maximization (4).

(iv) To judge the performance of the policy, $\mathscr{L}(v_i(t); \mathcal{I}_i(t))$ constitutes a metric for the difference between the target production and the realized one.

Proposition 1 (The MAP dynamics (long-memory)). *For each $t \geq 1$, the MAP estimator is*

$$v_i(t+1) = \begin{cases} \left(\dfrac{v_i^{(0)} + \gamma_i \sum_{\ell=1}^{t} z_i(\ell) s_i(\ell)}{(1 + \gamma_i \sum_{\ell=1}^{t} z_i(\ell)^2)} \right)^+, & \text{if } y_{i,j}(\ell)^{\phi\alpha_{i,j}} > 0 \text{ for all } j \in \mathcal{N},\ \ell \in \{1, \ldots, t\}, \\ v_i^{(*)} & \text{otherwise (continuous extension),} \end{cases} \tag{17}$$

where $\gamma_i = (\tau_i/\sigma_i)^2$ and the notation $(z)^+$ refers to $\max(0, z)$ for any $z \in \mathbb{R}$.

The first visible insight from (17) is that $v_i(t+1)$ is updated only based on the i-th material input decisions. Hence, the price taker behavior assumption (as adopted in Lemma 2) implies that firm i best response is independent from its knowledge of firm j realized production (where $j \neq i$). In other words, (17) demonstrates that $v_i(t)$ are conditionally independent once input decisions are taken and that the learning dynamics is invariant with respect to whether information of the market-wise equilibrium objects is assumed to be private or public.

The second noticeable insight is that in the case when there exists $j \in \mathcal{N}$ and $\ell \in \{1, \ldots, t\}$, such that $y_{i,j}(\ell)^{\phi\alpha_{i,j}} = 0$, the maximization of (16) admits infinite alternative solutions (all $v_i(t+1) \in \mathbb{R}_+$ are optimal for (16)). However, by continuous extension, $v_i(t+1)$ is determined in (17) from the case when $y_{i,j}(\ell)^{\phi\alpha_{i,j}} > 0$ for all $j \in \mathcal{N}$ and $\ell \in \{1, \ldots, t\}$. In such case, when input decisions are taken by a profit maximizing behaviour (4), this point of discontinuity can only happen if either $v_i(t) = 0$ or $p_i(t+1) = 0$ (which implies $y_{i,j}(t) = 0$ by Lemma 2), resulting in an absorbing state. To visualize the reason of such an absorbing state, it is sufficient to note that if there exists $j \in \mathcal{N}$ and $\ell \in \{1, \ldots, t\}$, such that $y_{i,j}(\ell)^{\phi\alpha_{i,j}} = 0$, then for the same j, there must also exist $\ell' \in \{1, \ldots, t+1\}$, such that $y_{i,j}(\ell') = 0$. In other words, by adopting the continuous extension to solve the indeterminacy of the MAP estimator, we obtain that *learning cannot be forgotten under the long-memory assumption.*

At this stage, it is worth reminding that the MAP estimator (17) is a random variable induced by the noisy production $x_i(t)$ through the idiosyncratic shocks $\varepsilon_i(t)$, at each time t. A fundamental step of the analysis of the learning dynamics is the evaluation of the expectation and variance of $v_i(t)$, as well as its convergence when t grows large.

Proposition 2 (Expectation and variance of firms' belief). *For each $1 \leq t \leq T-1$, let us define*

$$\overline{v}_i(\mathcal{Y}(t)) = \frac{v_i^{(0)} + \gamma_i v_i^{(*)} \tilde{z}_i^{(2)}(t)}{1 + \gamma_i \tilde{z}_i^{(2)}(t)}, \quad \text{and} \quad \overline{\varphi}_i(\mathcal{Y}(t)) = \frac{\gamma_i \sigma_i \tilde{z}_i^{(1)}(t)}{1 + \gamma_i \tilde{z}_i^{(2)}(t)}, \tag{18}$$

where $\tilde{z}_i^{(1)}(t) = \sum_{\ell=1}^{t-1} z_i(\ell)$, $\tilde{z}_i^{(2)}(t) = \sum_{\ell=1}^{t-1} z_i(\ell)^2$, and $\mathcal{Y}(t) = \{Y(\ell)\}_{\ell=1}^{t}$ As a consequence of Proposition 1, conditioned on the i-th input decisions up to period t, we have

$$\mathbb{E}[v_i(t+1)] = \overline{v}_i(\mathcal{Y}(t)) \tilde{F}_i(t) + |\overline{\varphi}_i(\mathcal{Y}(t))| \tilde{f}_i(t), \tag{19}$$

$$\mathbb{E}[v_i(t+1)^2] = \left(\overline{v}_i(\mathcal{Y}(t))^2 + |\overline{\varphi}_i(\mathcal{Y}(t))|^2 \right) \tilde{F}_i(t) + \overline{v}_i(\mathcal{Y}(t)) |\overline{\varphi}_i(\mathcal{Y}(t))| \tilde{f}_i(t), \tag{20}$$

where

$$\tilde{F}_i(t) \equiv G_i\left(\tilde{z}_i^{(1)}(t), \tilde{z}_i^{(2)}(t)\right) = 1 - F\left(- \left| \frac{\overline{v}_i(\mathcal{Y}(t))}{\overline{\varphi}_i(\mathcal{Y}(t))} \right| \right),$$

$$\tilde{f}_i(t) \equiv g_i(\tilde{z}_i^{(1)}(t), \tilde{z}_i^{(2)}(t)) = f\left(- \left| \frac{\overline{v}_i(\mathcal{Y}(t))}{\overline{\varphi}_i(\mathcal{Y}(t))} \right| \right).$$

Proposition 3 (Mode of firms' belief). *Let $\mathbb{M}[v_i(t) \mid \mathcal{Y}(t)]$ be the mode of the MAP estimator $v_i(t)$, conditioned on the i-th input decisions up to period t. Based on Proposition 1, we have*

$$\mathbb{M}[v_i(t+1) \mid \mathcal{Y}(t)] = \begin{cases} \overline{v}_i(\mathcal{Y}(t)) & \text{if } \dfrac{1}{|\overline{\varphi}_i(\mathcal{Y}(t))|\sqrt{2\pi}} \geq F\left(-\left|\dfrac{\overline{v}_i(\mathcal{Y}(t))}{\overline{\varphi}_i(\mathcal{Y}(t))}\right|\right), \\ 0 & \text{otherwise.} \end{cases}$$

This characterization of the expectation (Proposition 2) and mode (Proposition 3) of the MAP distribution allows for a closed-form analysis of its convergence, under the long memory assumption. This analysis is partitioned into five cases, depending on the dynamics of input decisions $y_{i,j}(t)$, as encoded by $z_i(t)$ (see notation (14)).[11]

Proposition 4 (Convergence of the expectation (long memory)). *The learning dynamics for the long memory method has five cases:*

(1) If for each $i \in \mathcal{N}$, the dynamics of input decisions satisfies $\lim_{t\to+\infty} |\tilde{z}_i^{(1)}(t)| = \lim_{t\to+\infty} \tilde{z}_i^{(2)}(t) = +\infty$ and $\lim_{t\to+\infty} \dfrac{v_i^{(0)} + \gamma_i v_i^{(*)} \tilde{z}_i^{(2)}(t)}{\gamma_i \sigma_i \tilde{z}_i^{(1)}(t)} = L < +\infty$, then we have

$$\lim_{t\to+\infty} \mathbb{E}[v_i(t)] = v_i^{(*)} \tilde{F}_i(t) + l\sigma_i \tilde{f}_i(t) \left(\lim_{\ell\to+\infty} \frac{|\tilde{z}_i^{(1)}(\ell)|}{\tilde{z}_i^{(2)}(\ell)} \right).$$

(2) If for each $i \in \mathcal{N}$, the dynamics of input decisions satisfies $\lim_{t\to+\infty} \tilde{z}_i^{(1)}(t) = L_1$ and $\lim_{t\to+\infty} \tilde{z}_i^{(2)}(t) = L_2$, then we have

$$\lim_{t\to+\infty} \mathbb{E}[v_i(t)] = \begin{cases} \dfrac{\left(v_i^{(0)} + \gamma_i v_i^{(*)} L_2\right) G_i(L_1, L_2) + \sigma_i \gamma_i L_1 g_i(L_1, L_2)}{1 + \gamma_i L_2} & \text{if } L_1 \neq 0, \\ \dfrac{v_i^{(0)} + \gamma_i v_i^{(*)} L_2}{1 + \gamma_i L_2} & \text{if } L_1 = 0. \end{cases}$$

(3) If for each $i \in \mathcal{N}$, the dynamics of input decisions satisfies $\lim_{t\to+\infty} \tilde{z}_i^{(1)}(t) = L_1 < +\infty$ and $\lim_{t\to+\infty} \tilde{z}_i^{(2)}(t) = \infty$, then we have

$$\lim_{t\to+\infty} \mathbb{E}[v_i(t)] = v_i^{(*)}.$$

(4) If for each $i \in \mathcal{N}$, the dynamics of input decisions satisfies $\lim_{t\to+\infty} \tilde{z}_i^{(1)}(t) = \infty$ and $\lim_{t\to+\infty} \tilde{z}_i^{(2)}(t) = L_2 < +\infty$, then we have

$$\lim_{t\to+\infty} \mathbb{E}[v_i(t)] = +\infty.$$

(5) If for each $i \in \mathcal{N}$, the dynamics of input decisions satisfies $\lim_{t\to+\infty} |\tilde{z}_i^{(1)}(t)| = \lim_{t\to+\infty} \tilde{z}_i^{(2)}(t) = \lim_{t\to+\infty} \dfrac{v_i^{(0)} + \gamma_i v_i^{(*)} \tilde{z}_i^{(2)}(t)}{\gamma_i \sigma_i \tilde{z}_i^{(1)}(t)} = +\infty$, then we have

$$\lim_{t\to+\infty} \mathbb{E}[v_i(t)] \text{ is undetermined.}$$

[11] Although not explicitly formalized, this time limit implies an equivalent limit of the length of the time horizon T.

This proposition reveals that under the long memory assumption, there are two possible dynamics of input decisions in which the expected belief of the returns to scale converges to $v_i^{(*)}$: case (1) and case (3). The first one encompasses situations in which there exists $j \in \mathcal{N}$ and $\ell \in \{1, \dots, t\}$ for which $y_{i,j}(\ell)^{\alpha_{i,j}} = 0$, and it is the result of the convention to characterize the discontinuity point by a continuous extension, as established in Proposition 1. Following this reasoning, by the profit maximizing behaviour characterized in Proposition 2, this discontinuity point is only possible if either $v_i(t) = 0$ or $p_i(t+1) = 0$. Thus, case (1) has little economic interest.

Contextually, case (3) requires $\lim_{t\to+\infty} \tilde{z}_i^{(2)}(t) = \infty$, which can only happen if the path of input decisions is such that:

$$\lim_{t\to+\infty} \prod_{j=1}^{n} (y_{i,j}(t))^{\phi\alpha_{i,j}} \neq 1.$$

Since $\lim_{t\to+\infty} \tilde{z}_i^{(1)}(t) < +\infty$, case (3) requires that there exists $t_0 \in \mathbb{N}$, such that $\prod_{j=1}^{n} (y_{i,j}(t))^{\phi\alpha_{i,j}}$ oscillates around one for all $t > t_0$, which is also of relatively minor economic significance.

Conversely, the most economically meaningful case is (2), which is associated to a converging pattern of input decisions, and which results to a biased estimator.

Overall, a summary interpretation of this convergence dynamics is that the MAP estimator is generally biased (its limit expectation differs from the true parameter value $v_i^{(*)}$), except for very specific and unlikely cases of input decisions. However, an economically more favourable conclusion can be reached when focusing on the mode of the estimator. The next proposition shows that cases (3) and (4) of Proposition 4 (for which the expected belief converges to $v_i^{(*)}$ or diverges to $+\infty$, respectively) are both associated to a correct mode of the MAP estimator.

Proposition 5 (Convergence of the mode (long memory))**.** *Let $v_i(0) \geq 0$ and $v_i^{(*)} > 0$, for all $i \in \mathcal{N}$. Let us define the ordered set*

$$\overline{\Psi}_i = \mathbb{N} / \left\{ t \in \mathbb{N} \ : \ |\overline{\varphi}_i(\mathcal{Y}(t))| \sqrt{2\pi} F\left(- \left| \frac{\overline{v}_i(\mathcal{Y}(t))}{\overline{\varphi}_i(\mathcal{Y}(t))} \right| \right) > 1 \right\}, \tag{21}$$

and index its elements by $\varrho(t)$, for $t \in \mathbb{N}$. There exists a sub-sequence $\{\overline{v}_i(\varrho(t))\}_t$ such that

$$\lim_{t\to+\infty} \mathbb{M}[v_i(t) \mid \mathcal{Y}(t)] = \begin{cases} 0 & \text{if } \overline{\Psi}_i \text{ is a finite set,} \\ v_i^{(*)} & \text{if } \overline{\Psi}_i \text{ is an infinite set and } \lim_{t\to+\infty} z_i(t) = +\infty, \\ \dfrac{v_i^0 + \gamma_i v_i^{(*)} l}{1 + \gamma_i l} & \text{if } \overline{\Psi}_i \text{ is an infinite set and } \lim_{t\to+\infty} z_i(t) = L < +\infty. \end{cases}$$

By means of explanation, the ordered set $\overline{\Psi}_i$ is an auxiliary representation of the collection of time periods for which $\mathbb{M}[v_i(t) \mid \mathcal{Y}(t)] > 0$. This allows selecting a sub-sequence to study the convergence of the original sequence, as detailed in Appendix A. Therefore, the mode of the MAP estimator always converges, even in cases when its expectation doesn't. The most interesting case is when $z_i(t)$ diverges to $+\infty$.

Corollary 1. *We have the following implication:*

$$\text{if} \quad \lim_{t\to+\infty} \prod_{j=1}^{n} (y_{i,j}(t))^{\phi\alpha_{i,j}} = +\infty \quad \text{then} \quad \lim_{t\to+\infty} \mathbb{M}[v_i(t) \mid \mathcal{Y}(t)] = v_i^{(*)}.$$

An insightful interpretation of this corollary is that, under the long-memory approach, substantial and enduring economic growth, coupled with a consequent rise in the series of input quantities, enhances the dynamic learning of the true returns to scale (due to the divergence of $z_i(t)$). This establishes a theoretical relationship between learning and economic growth, which has been already conjectured since the seminal work of Arrow (1962) and extensively elucidated by Solow (1997).

As announced in Section 3.3, Propositions 1-5 confirm that the proposed learning mechanism allows for a high level of generality, as the analysed theoretical properties and implications are valid for any characterization of the input decisions and can still be adopted when market players take sub-optimal decisions, based on heuristic algorithms or rules of thumb (Blonski, 1999). This is explicitly revealed in the MAP estimator (17), whose functional form remains invariant for any characterization of $Y(t)$ (through the reparametrization (14)).

4.2 Short memory learning

Focusing on a short-memory method, we consider a point-wise estimation approach constructed by enforcing the following assumptions on the information set and the initial knowledge.

Assumption 4. *At period t, firm's i information set is given by the last realized production: $\mathcal{I}_i(t-1) = \{x_i(t-1)\}$.*

Assumption 5. *At each period t, firms re-initialize their probability distribution $\pi_{t,i}(v_i)$ as a zero-truncated Gaussian with parameters $v_i(t-1)$ and τ_i.*

In this learning approach, the Bayesian update only keeps track of one previous period, by re-initializing the prior distribution to a truncated Gaussian centered at the previous MAP estimation. This captures a memory loss, where the information of all previous periods is encoded in the previous update $v_i(t-1)$. Hence, the short memory method departs from Assumption 2 and Assumption 3, and replace them with Assumption 4 and Assumption 5, respectively.

The likelihood function of $v_i(t)$ reduces to the probability distribution over the period $t-1$ production, which is induced by the idiosyncratic shock $\eta_i(t-1)$:

$$\mathscr{L}(v_i(t); \mathcal{I}_i(t-1)) = \frac{1}{(2\pi\sigma_i)^{\frac{1}{2}}} \exp\left(-\frac{1}{2\sigma_i^2} \left(\log x_i(t-1) - \log \mu_i(t-1)\right)^2\right), \qquad \text{for } 2 \leq t \leq T. \quad (22)$$

The general form of the Bayesian posterior distribution of $v_i(t)$ at the t-th period is still (16) and (17). With few algebrical changes, we obtain the MAP estimator and its expectation and variance, in line with Propositions 1 and 2.

Proposition 6 (The MAP dynamics (short-memory))**.** *For each $t \geq 1$, the MAP estimator is*

$$v_i(t+1) = \begin{cases} \left(\dfrac{v_i(t) + \gamma_i z_i(t)\left(\sigma_i \varepsilon_i(t) + v_i^{(*)} z_i(t)\right)}{1 + \gamma_i z_i(t)^2}\right)^+, & \textit{if } y_{i,j}(t)^{\phi\alpha_{i,j}} > 0 \textit{ for all } j \in \mathcal{N}, \\ v_i^{(*)} & \textit{otherwise (continuous extension).} \end{cases} \quad (23)$$

Proposition 7 (Expectation and variance of firms' belief)**.** *For each $1 \leq t \leq T-1$, let us define*

$$\overline{\overline{v}}_{i,t}(\mathcal{Y}(t)) = \frac{v_i(t) + \gamma_i v_i^{(*)} z_i(t)^2}{1+\gamma_i z_i(t)^2}, \qquad \text{and} \quad \overline{\overline{\varphi}}_i(\mathcal{Y}(t)) = \frac{\gamma_i \sigma_i z_i(t)}{1+\gamma_i z_i(t)^2}.$$

Conditioned on the i-th input decisions up to period $t-1$, we have

$$\mathbb{E}[v_i(t+1)] \;=\; \overline{\overline{v}}_i(\mathcal{Y}(t))\tilde{\tilde{F}}_i + |\overline{\overline{\varphi}}_i(\mathcal{Y}(t))|\tilde{\tilde{f}}_i, \tag{24}$$

$$\mathbb{E}[v_i(t+1)^2] \;=\; \left(\overline{\overline{v}}_i(\mathcal{Y}(t))^2 + |\overline{\overline{\varphi}}_i(\mathcal{Y}(t))|^2\right)\tilde{\tilde{F}}_i + \overline{\overline{v}}_i(\mathcal{Y}(t))|\overline{\overline{\varphi}}_i(\mathcal{Y}(t))|\tilde{\tilde{f}}_i, \tag{25}$$

where

$$\begin{aligned}\tilde{\tilde{F}}_i &\equiv G_i(z_i(t), z_i(t)^2) = 1 - F\left(-\left|\frac{\overline{\overline{v}}_i(\mathcal{Y}(t))}{\overline{\overline{\varphi}}_i(\mathcal{Y}(t))}\right|\right), \\ \tilde{\tilde{f}}_i &\equiv g_i(z_i(t), z_i(t)^2) = f\left(-\left|\frac{\overline{\overline{v}}_i(\mathcal{Y}(t))}{\overline{\overline{\varphi}}_i(\mathcal{Y}(t))}\right|\right).\end{aligned}$$

In line with Propositions 4 and 5, an analogous analysis of the learning dynamics is performed hereafter for the short-memory method.

Proposition 8 (Mode of firms' belief)**.** *Let $\mathbb{M}[v_i(t) \mid \mathcal{Y}(t)]$ be the mode of the MAP estimator $v_i(t)$, conditioned on the i-th input decisions up to period t. Based on Proposition 6, we have*

$$\mathbb{M}[v_i(t+1) \mid \mathcal{Y}(t)] = \begin{cases} \overline{\overline{v}}_i(\mathcal{Y}(t)), & \text{if } \dfrac{1}{|\overline{\overline{\sigma}}_i(\mathcal{Y}(t))|\sqrt{2\pi}} \geq F\left(-\left|\dfrac{\overline{\overline{v}}_i(\mathcal{Y}(t))}{\overline{\overline{\varphi}}_i(\mathcal{Y}(t))}\right|\right) \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 9 (Convergence of the expectation (short memory))**.** *We have the following convergence of the learning dynamics*

$$\lim_{t\to+\infty} \mathbb{E}_{\varepsilon_i(t)}[v_i(t)] = \begin{cases} v_i^{(*)} & \text{if } \lim\limits_{t\to+\infty} z_i(t) \neq 0 \\ v_i^{(*)} + \dfrac{v_i(0) - v_i^{(*)}}{\prod\limits_{h=1}^{\infty}(1+\gamma_i z_i(h)^2)} & \text{if } \lim\limits_{t\to+\infty} z_i(t) = 0. \end{cases}$$

Proposition 10 (Belief convergence (short memory))**.** *Let us define the ordered set*

$$\Psi_i = \mathbb{N} / \left\{ t \in \mathbb{N} \;:\; |\overline{\overline{\varphi}}_i(\mathcal{Y}(t))|\sqrt{2\pi} F\left(-\left|\frac{\overline{\overline{v}}_i(\mathcal{Y}(t))}{\overline{\overline{\varphi}}_i(\mathcal{Y}(t))}\right|\right) > 1 \right\}, \qquad \text{for } i \in \mathcal{N},$$

along with the order function $\psi_i(t)$ that assigns to each $t \in \mathbb{N}$ the element of Ψ_i in the t-th position. We assume $v_i(0) \geq 0$ and $v_i^{(*)} > 0$, for all $i \in \mathcal{N}$, and consider the indicator function

$$\tilde{\mathbb{1}}_{i,t-1} = \begin{cases} 1 & \text{if } t \in \Psi_i, \\ 0 & \text{otherwise.} \end{cases}$$

Conditioned on the sequence of material input decisions $\mathcal{Y}(t)$, the MAP estimator follows a deterministic sequence $\{\overline{v}_i(t)\}_{t\in\Psi}$, where

$$\overline{v}_i(t+1) = v_i(t)\left(\prod_{h=0}^{t-1} \frac{\tilde{\mathbb{1}}_{i,h}}{(1+\gamma_i z_i(h)^2)}\right) + v_i^{(*)}\left(\gamma_i \sum_{h=0}^{t-1} z_i(h)^2 \prod_{s=h}^{t-1} \frac{\tilde{\mathbb{1}}_{i,s}}{(1+\gamma_i z_i(s)^2)}\right). \tag{26}$$

We claim that there exists a sub-sequence $\{\overline{v}_i(\psi_i(t))\}_t$ such that

$$\lim_{t\to+\infty} \overline{v}_i(\psi_i(t)) = \begin{cases} 0 & \text{if } \Psi_i \text{ is a finite set,} \\ v_i^{(*)} & \text{if } \Psi_i \text{ is an infinite set and } \lim\limits_{t\to+\infty} z_i(t) \neq 0, \\ v_i^{(*)} + \dfrac{v_i(\psi_i(0)) - v_i^{(*)}}{\prod\limits_{h=1}^{\infty} (1+\gamma_i z_i(\psi_i(h))^2)} & \text{if } \Psi_i \text{ is an infinite set and } \lim\limits_{t\to+\infty} z_i(t) = 0. \end{cases}$$

While mirroring the convergence analysis of Propositions 4 and 5 for the long memory method, Propositions 9 and 10 require less restrictive conditions. In particular, Proposition 10 provides evidence of the fact that the learning dynamics converges to $v_i^{(*)}$, independently from how material input decisions are taken, when $\gamma_i = (\tau_i/\sigma_i)^2$ grows large or $\{z_i(t)\}_t$ does not converge to zero.

It is worth mentioning that these theoretical figures integrate some contextual results about the convergence pattern of a Bayesian learning dynamics in repeated games (Nyarko, 1994, 1998), while establishing the advantages and disadvantages of the MAP estimator to infer the unknown returns to scale in multi-sector economies.

4.3 The discontinuity point of the maximum a-posteriori estimation

Beside the independence and convergence properties of the MAP estimator studied in Subsection 4.1 and Subsection 4.2, one downside for firm i to use such an estimation method at time t is related to its discontinuity when there exists $j \in \mathcal{N}$ and $\ell \in \{1, \dots, t\}$, such that $y_{i,j}(\ell)^{\phi\alpha_{i,j}} = 0$ (for the long-memory method), or when there exists $j \in \mathcal{N}$, such that $y_{i,j}(t)^{\phi\alpha_{i,j}} = 0$ (for the short-memory method). In these cases, the maximization of (16) or (22), for the long and short memory respectively, admits infinite alternative solutions. In Proposition 1 and Proposition 6, we adopt the continuous extension for the characterization of the MAP estimator at this point of discontinuity.

Hereafter, the inquiries pertain to the likelihood of the analyzed dynamic system passing through this particular state. In fact, we have previously mentioned that, when input decisions are taken by a profit maximizing behaviour (4), this point of discontinuity can only happen if either $v_i(t) = 0$ or $p_i(t+1) = 0$ (which implies $y_{i,j}(t) = 0$ by Lemma 2).

Proposition 11 (Probability of $v_i(t) = 0$ (long memory))**.** *Under Assumption 2 and Assumption 3, we have that for each $t \geq 1$,*

$$\mathbb{P}\left(v_i(t) = 0 \mid \mathcal{I}_i(t-1)\right) = F\left(-\frac{\tau_i v_i^{(*)}}{\sigma_i^3} - \frac{v_i^{(0)} + m_i \sum_{\ell=1}^{t-1} z_i(\ell)}{\sigma_i \sum_{\ell=1}^{t-1} z_i(\ell)^2}\right).$$

Hence,

$$\begin{cases} \lim\limits_{\sigma_i\to 0} \mathbb{P}\left(v_i(t) = 0 \mid \mathcal{I}_i(t-1)\right) & = 0, \\ \lim\limits_{\tau_i\to\infty} \mathbb{P}\left(v_i(t) = 0 \mid \mathcal{I}_i(t-1)\right) & = 0. \end{cases}$$

where F is the probability distribution function of a standardized Gaussian random variable.

Proposition 12 (Probability of $v_i(t) = 0$ (short memory))**.** *Under Assumption 4 and Assumption 5, we have that for each $t \geq 1$,*

$$\mathbb{P}\left(v_i(t) = 0 \mid \mathcal{I}_i(t-1)\right) = F\left(-\frac{\tau_i v_i^{(*)}}{\sigma_i^3} - \frac{v_i(t-1) + m_i z_i(t-1)}{\sigma_i z_i(t-1)^2}\right).$$

Hence

$$\begin{cases} \lim_{\sigma_i \to 0} \mathbb{P}\left(v_i(t) = 0 \mid \mathcal{I}_i(t-1)\right) & = 0, \\ \lim_{\tau_i \to \infty} \mathbb{P}\left(v_i(t) = 0 \mid \mathcal{I}_i(t-1)\right) & = 0. \end{cases}$$

The overall picture of the MAP estimator to infer the returns to scale parameter suggests that a major role is plaid by the variance of the idiosyncratic productivity shocks σ_i and the degree of uncertainty of the a-priory distribution τ_i. In fact, for any $t \geq 1$, the conditional probability of $v_i(t) = 0$ (which implies $y_{i,j}(t) = 0$, for all $j \in \mathcal{N}$ in the next production period) is negligible when the variance of the idiosyncratic productivity shock is small and the uncertainty of the a-priory distribution is large. This suggests an adequate economic context in which the proposed MAP estimator can be fruitfully applied, but also the main drawback induced by large idiosyncratic fluctuations.

5 The price of incorrect knowledge

After uncovering how information collection and processing unfolds through the multi-sector economy based on input decisions, the reverse effect of $v_i(t)$ on (11) and (13) remains to be investigated. In this line, a well-established stream of literature has tried to disentangle the sources of production input misallocation (David and Venkateswaran, 2017; Jones, 2011; Hsieh and Klenow, 2009) by looking at technological frictions and firm-specific factors. Since the incorrect knowledge of the returns to scale of production can be included within this debate, this section studies how this inferential bias translates into incorrect input decisions and how a price mechanism might mitigate input misallocations.

While in the previous sections no assumptions have been made on $v_i^{(*)}$, to deduce stylized facts, we focus hereafter on the limit behaviour of market-wise equilibrium objects under the constant returns to scale assumption (i.e., $v_i^{(*)} = 1$, for $i \in \mathcal{N}$).[12] We introduce the notation:

$$\tilde{\mathbf{h}} = \begin{bmatrix} -\frac{\log q_1}{\phi} - \sum_{h=1}^n \alpha_{1,h} \log \alpha_{1,h} \\ \vdots \\ -\frac{\log q_n}{\phi} - \sum_{h=1}^n \alpha_{n,h} \log \alpha_{n,h} \end{bmatrix} \quad \text{and} \quad \hat{r}_i(t,v) = p_i(t+1) q_i(l_i(t))^{(1-\phi)} \left(\prod_{j=1}^n (y_{i,j}(t))^{\phi \alpha_{i,j}} \right)^v.$$

Further, we consider a generalized form of the Bonacich's beta-centrality vector, which we define as $\beta_j(\alpha, A, \mathbf{h}) = (I - \alpha A)^{-1}\mathbf{h}$, for any arbitrary $\alpha \in \mathbb{R}_+$, $\mathbf{h} \in \mathbb{R}^n$ and $A \in \mathbb{R}_+^{n \times n}$,[13] and we provide

[12] The constant returns to scale is the underlying hypothesis of Acemoglu et al. (2012, 2017). While we do not confine ourselves to this assumption when studying the learning dynamics, the particular case of constant returns to scale is used in this section to highlight the impact of the incorrect knowledge on the limit behaviour of prices.

[13] For a given square matrix A, the Bonacich's beta-centrality vector is defined as $\boldsymbol{\beta}(\alpha, A) = \sum_{l=1}^{\infty} \alpha^{l-1} A^l \mathbf{1} = (I - \alpha A)^{-1}\mathbf{1}$, where $\mathbf{1}$ is a vector of ones. The generalized Bonacich's beta-centrality vector coincides with the classical Bonacich's beta-centrality vector when $\mathbf{h} = \mathbf{1}$. While classical Bonacich's beta-centrality corresponds to a weighted sum of paths connecting to other industries, this generalized Bonacich's beta-centrality vector, weights the contribution of the different industries using the exogenous information provided by vector $\mathbf{h}$. For more details on the Bonacich's beta-centrality vector see Bonacich (2007).

a characterization of equilibrium prices (Lemma 3) and their limit behaviour under the constant returns to scale condition (Proposition 13).

Lemma 3 (Equilibrium prices)**.** *Let us define*

$$\overline{D}_i(t) = q_i(1-\phi)^{1-\phi}\left(\phi v_i \prod_{j=1}^{n}\left(\frac{\alpha_{i,s}}{p_s(t)}\right)^{\alpha_{i,s}}\right)^{\phi v_i}, \qquad \text{for } t = 1,\dots,T-1.$$

We have the following equilibrium prices and expected production:

$$p_i(t+1) = \frac{w(t+1)^{1-\phi}}{\overline{D}_i(t)}\left(\frac{v_i\phi}{\sum_{j=1}^{n}\left[A^{-1}\right]_{j,i}\left(p_j(t)x_j(t-1) - \overline{w}(t)\kappa_j\right)}\right)^{\phi(v_i-1)}, \tag{27}$$

$$\mu_i(t) = \frac{\overline{D}_i(t)}{w(t+1)^{1-\phi}}\left(\frac{\sum_{j=1}^{n}\left[A^{-1}\right]_{j,i}\left(p_j(t)x_j(t-1) - \overline{w}(t)\kappa_j\right)}{v_i\phi}\right)^{1+\phi(v_i-1)}, \tag{28}$$

for each $i \in \mathcal{N}_+$ (where $[A^{-1}]_{j,i}$ denotes the (j,i) element of matrix A^{-1}) and

$$p_i(t+1) = d_i \quad \text{where } d_i \text{ is the solution of } \sum_{i\in\mathcal{N}_0}\left(\frac{(1-\phi)d_i}{w(t+1)}\right)^{\frac{1}{\phi}} = 1 - \sum_{i\in\mathcal{N}_+} l_i(t), \tag{29}$$

$$\mu_i(t) = \left(\frac{(1-\phi)p_i(t+1)}{w(t+1)}\right)^{\frac{1-\phi}{\phi}}, \tag{30}$$

for each $i \in \mathcal{N}_0$.

Proposition 13 (Price convergence)**.** *Let us assume that $\{v_i(t)\}_t$ converges, that $w(t+1)/w(t)$ is uniformly bounded,*[14] *and that $v_i^{(*)} = 1$, for all $i \in \mathcal{N}$. We define the sequence*

$$W(t) = \log\left(\frac{w(t)}{w(t-1)}\right) + \phi W(t-1).$$

A necessary condition for the incorrect knowledge of v_i not to translate into the expected revenue (namely, $\hat{r}_i(t, v_i(t)) = \hat{r}_i(t, v_i^{(*)})$), is that the limit equilibrium prices (27) *to be proportional to the generalized Bonacichs beta-centrality vector:*

$$p_i(t+1) \propto w(t+1)\exp\left(\phi\beta_i\left(\phi, A, \tilde{\mathbf{h}}\right) - \phi W(t)\right). \tag{31}$$

Proposition 13 uncovers a price mechanism which is consistent not only with the correct knowledge of the returns to scale $v_i^{(*)}$, but also with any $v_i(t)$, as long as $\{v_i(t)\}_t$ converges. Hence, for the nominal revenue of each sector not to be affected by the inferential bias, the limit prices must reflect the centrality of sectors in the multi-sector network. This condition encompasses cases in which the limit prices are log-linear in the generalized Bonacich's beta-centrality of the input-output elasticity structure and their limit variation rates are finite and uniform for all sectors.[15] As a corollary, if $w(t)$ converges, all prices also do.

[14] The uniform boundedness implies that there exists a constant $\xi > 0$ such that for each $t > t_0$, we have $w(t+1) \leq \xi w(t)$, for some $t_0 \in \mathbb{N}$.

[15] Note that the convergence condition in Proposition 13 only requires $\{v_i(t)\}_t$ to converge to an arbitrary point, not necessarily $v_i^{(*)}$.

A further qualitative interpretation of this result is that the inferential bias on the returns to scale happens to be internalized in the price mechanism. An analogous prices dynamics has been studied by Acemoglu et al. (2012, 2017), providing an analytical relationship between prices and the influence vector, which they define to mirror the Bonacich centrality vector corresponding to the inter-sectoral network.

It remains to investigate the existence of equilibrium prices that satisfy the limit behaviour (31). This is established in the next proposition.

Proposition 14 (The impact of incorrect belief)**.** *Let us assume that $\{v_i(t)\}_t$ converges to v_i, with $v_i \neq v_i^{(*)}$, for all $i \in \mathcal{N}$. Let $M = ((I - \phi VA)(I - \phi A)^{-1} + I)$, with V being a diagonal matrix with elements $v_1, \ldots, v_n$. Equilibrium prices satisfy* (31) *if and only if there exists t_0, such that for all $t > t_0$ the following fixed-point equation admits a solution in $\mathbb{R}_+$:*

$$w(t+1) = \frac{1-\phi}{\phi} \sum_{i=1}^{n} \frac{1}{w(t+1)^{\frac{1}{v_i - 1}}} \iota_i\Big(\{w(h)\}_{h=1}^{t-1}\Big), \tag{32}$$

where v_i is the limit point of $\{v_i(t)\}_t$, and $\iota_i(\{w(h)\}_{h=1}^{t})$ only depends on past salaries:

$$\iota_i(\{w(h)\}_{h=1}^{t}) = \exp\left(\frac{\left[M\tilde{\mathbf{h}}\right]_i - \hat{\phi} + v_i(\log\phi + \tilde{\phi})}{1 - v_i}\right)\left(\frac{w(t)^{v_i+1}}{w(t-1)}\right)^{\frac{1}{v_i-1}} \prod_{h=1}^{t-2}\left(\frac{w(h)}{w(h+1)}\right)^{\phi^{t-1-h}},$$

and $\hat{\phi} = -\log(1-\phi) - \frac{\log(\phi)}{1-\phi}$.

Note that by construction, if a solution of (32) exists, it also satisfies the market clearing conditions (5) and (6). While this equation can be computationally solved by state-of-the-art numerical procedures, for its closed-form analysis, we refer to Smart (1974). The following proposition establishes a sufficient condition for the existence of a fixed point of (32).

Proposition 15 (Fixed point)**.** *We have the following result:*

- *if for each $i \in \mathcal{N}$, $v_i > v_i^{(*)} = 1$, then* (32) *admits a unique fixed-point such that $\omega(t+1) > 0$;*
- *if for each $i \in \mathcal{N}$, $v_i < v_i^{(*)} = 1$, then the unique fixed-point of* (32) *is $\omega(t+1) = 0$.*

By means of explanation, when the limit point of $\{v_i(t)\}_t$ is such that $i \in \mathcal{N}$, $v_i > v_i^{(*)}$ (firms are overestimating their returns to scale), (32) admits a unique and non-trivial fixed-point, which entails that the price dynamics (31) (i.e., the only price dynamics compatible with $\hat{r}_i(t, v_i(t)) = \hat{r}_i(t, v_i^{(*)})$) satisfies the market clearing conditions. Together with the convergence analysis in Propositions 4 and 5 (for the long-memory method) and Propositions 9 and 10 (for the short-memory method), this result provides a positive assessment to the auspicious conjecture that input missallocation (which translates into an erroneous estimation of the expected revenue) can be asympthotically circumvented or mitigated by a learning mechanism and by a pricing/salary mechanism that favors this convergence. Although this result is only valid under the restrictive case $v_i^{(*)} = 1$, the existence of equilibrium prices compatible with $\hat{r}_i(t, v_i(t)) = \hat{r}_i(t, v_i^{(*)})$ under this specific condition opens the possibility of studying further conditions under which the incorrect knowledge of profit-relevant parameters can be mitigated by the price mechanism.

6 Statistical inference of the learning parameters

With a view to providing an empirical grounding for our theoretical results, this section proposes a calibration procedure for the learning parameters of real intersectoral networks.

Mirroring the traditional approach from Zellner et al. (1966), we tailor a least square estimation based on the equilibrium equations in Lemma 2 and the learning equations in Proposition 1, that can be fitted through available data. Specifically, we use the detailed benchmark input-output accounts spanning the 1999-2019 period from the Bureau of Economic Analysis. These data constitute the finest level of disaggregation available for the intersectoral input-output network in the U.S, with most sectors (roughly) corresponding to four-digit SIC codes. Based on the commodity-by-commodity tables, we denote with $u_{i,j}(t)$ its (i,j) entry at year t. This contains the value of spending on commodity i per dollar of production of commodity j evaluated at current producer prices. Using our notation, we obtain

$$\begin{aligned} u_{i,j}(t) &= \frac{y_{j,i}(t)p_i(t)}{x_j(t)p_j(t+1)}, \\ \log u_{j,i}(t) &= \log \frac{y_{i,j}(t)p_j(t)}{x_i(t)p_i(t+1)} = \log \frac{y_{i,j}(t)p_j(t)}{\hat{r}_i(t)} - \log \frac{\eta_i(t)}{q_i} = \log \phi v_i(t)\alpha_{i,j} - \sigma_i\varepsilon_i(t). \end{aligned} \tag{33}$$

Next, from the Bureau of Economic Analysis we collect information about the value added by sector and the chain-type price indexes for intermediate inputs $\tilde{p}_i(t)$, for each sector $i \in \mathcal{N}$, corresponding to the same four-digit SIC codes. This can be used as a proxy for $r_j(t) = x_j(t)p_j(t+1) = \hat{r}_i(t)\frac{\eta_i(t)}{q_i}$ and $p_i(t)$ (once $p_i(0)$ is known, so that $p_i(0)\tilde{p}_i(t)$). Fixing these observable quantities, $z_i(t)$ can be established as a function of the unknown A, ϕ and $\mathbf{p}(0)$ only:

$$Z_i(A, \mathbf{p}(0)) := z_i(t) = \log \left(\prod_j (y_{i,j}(t))^{\phi\alpha_{i,j}} \right) = \phi \sum_{j=1}^{n} \alpha_{i,j} \left(\log \tilde{u}_{j,i}(t) - \log p_j(0)\tilde{p}_j(t) \right),$$

where $\tilde{u}_{j,i}(t)$ denotes an observable quantity encoding $\tilde{u}_{j,i}(t) = \log(u_{j,i}(t)\hat{r}_j(t)\eta_i(t)/q_i)$. This allows building a proxy for $v_i(t)$ in (17) and (23) (that we denote as $\tilde{V}_i^{LM}(t)$ and $\tilde{V}_i^{SM}(t)$ for the long and short memory, respectively), that depends on n^2+3n+2 unknown quantities: γ_i, ϕ, $p_1(0), \ldots, p_n(0)$, $v_1^{(0)}, \ldots, v_n^{(0)}$, $v_1^{(*)} \ldots, v_n^{(*)}$, and A. In the long memory case, this is defined as follows:

$$\tilde{V}_i^{LM}(t) = \frac{v_i^{(0)} + \gamma_i v_i^{(*)} \sum_{\ell=1}^{t} \phi^2 \left(\sum_{j=1}^{n} \alpha_{i,j}\tilde{u}_{j,i}(\ell) - \log p_j(0)\tilde{p}_j(\ell) \right)^2}{1 + \gamma_i \sum_{\ell=1}^{t} \phi^2 \left(\sum_{j=1}^{n} \alpha_{i,j}\tilde{u}_{j,i}(\ell) - \log p_j(0)\tilde{p}_j(\ell) \right)^2}. \tag{34}$$

In the short memory case, this is defined as follows:

$$\tilde{V}_i^{SM}(t) = \frac{\tilde{V}_{i,t-1}^{SM} + \gamma_i v_i^{(*)} \phi^2 \left(\sum_{j=1}^{n} \alpha_{i,j}\tilde{u}_{j,i}(t) - \log p_j(0)\tilde{p}_j(t) \right)^2}{1 + \gamma_i \phi^2 \left(\sum_{j=1}^{n} \alpha_{i,j}\tilde{u}_{j,i}(t) - \log p_j(0)\tilde{p}_j(t) \right)^2}, \tag{35}$$

Combining (33) and (11), we are able to build the error function after observing a sequence of T commodity-by-commodity tables:

$$
\begin{aligned}
\mathcal{E}(\hat{\gamma}, \hat{\mathbf{p}}, \mathbf{q}, \mathbf{v}^{(0)}, \mathbf{v}^{(*)}, A) &= \sum_{t=1}^{T} \left[\sum_{(i,j):u_{j,i}(t)>0} \left(\log \tilde{u}_{j,i}(t) - \log q_i \phi \tilde{V}_i^f(t) \alpha_{i,j} \right)^2 \right], \\
&= \sum_{t=1}^{T} \left[\sum_{(i,j):u_{i,j}(t)>0} \left(\log \frac{\tilde{u}_{j,i}(t)}{(q_i \phi \alpha_{i,j})} \frac{(1+\gamma_i R_i(A, \hat{\mathbf{p}})^2)}{v_i^{(0)} + \gamma_i v_i^{(*)} R_i(A, \hat{\mathbf{p}})^2} \right)^2 \right],
\end{aligned}
$$

where $f \in \{LM, SM\}$. The least-square estimation is obtained by solving

$$
\min \mathcal{E}(\hat{\gamma}, \hat{\mathbf{p}}, \mathbf{v}^{(0)}, \mathbf{v}^{(*)}, A), \quad \text{subj. to} \ \sum_{j=1}^{n} \alpha_{i,j} = 1, \ \hat{\gamma}, \hat{\mathbf{p}}, \mathbf{v}^{(0)}, \mathbf{v}^{*} \geq 0. \tag{36}
$$

Using this estimation approach and benefiting from the available observations of $\tilde{u}_{i,j}(t)$, $r_i(t)$, and $\tilde{p}_i(t)$, we solve (36) for both cases of long and short learning methods. Figure 1 shows the estimated input-output elasticity structures $\alpha_{i,j}$ in the form of network plots for both cases. Figure 2 illustrates the corresponding frequency values of $\alpha_{i,j}$ in the form of histograms.

Figure 1: Network plots for the estimated input-output elasticity structures, based on the long memory assumption (left plot) and short memory assumption (right plot).

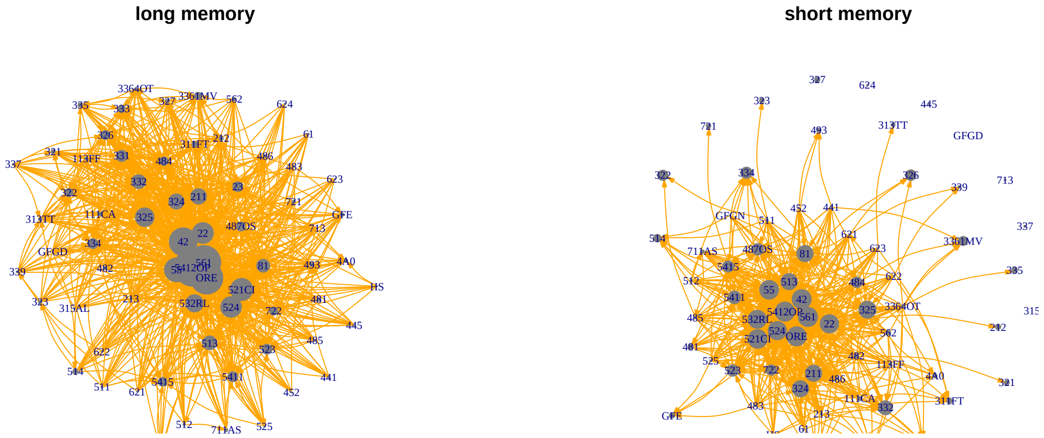
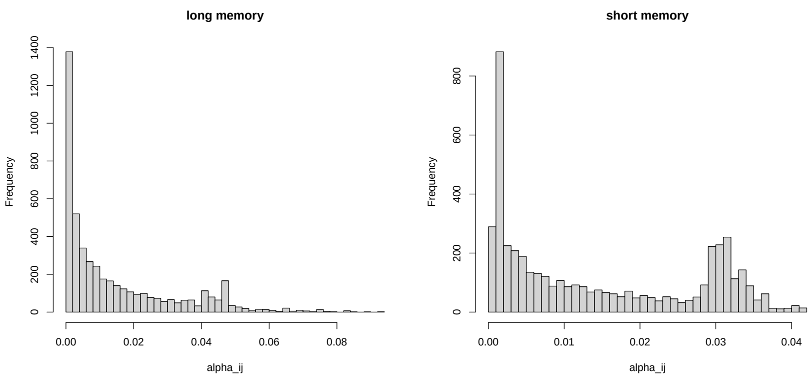


Figure 2: Histograms for the estimated input-output elasticity structures, based on the long memory assumption (left plot) and short memory assumption (right plot).



Exploring the estimated values of $v_1^{(0)}, \ldots v_n^{(0)}$ and $v_1^{(*)}, \ldots v_n^{(*)}$, Table 1 provides a classification of *upward* and *downward* learning industries, based on $v_i^{(0)} < v_i^{(*)}$ and $v_i^{(0)} \geq v_i^{(*)}$, respectively. We regard an industry to be upward (downward) learning if its starting belief of the returns to scale is smaller (larger) than the actual returns to scale.

Table 1: Classification of the upward versus downward learning industries based on the estimated returns to scale parameters $v_i^{(0)}$ and $v_i^{(*)}$.

	$v_i^{(0)} < v_i^{(*)}$ (upward learning)	$v_i^{(0)} \geq v_i^{(*)}$ (downward learning)
Long memory	Support activities for mining, Mineral products, Primary metals, Motor vehicles, Furniture, Paper products, Wholesale trade, Food and beverage stores, General merchandise stores, Other retail, Rail transportation, Water transportation, Truck transportation, Transit and ground transportation, Other transportation activities, Warehousing and storage, Internet publishing and information services, Securities and commodity contracts, Insurance carriers, Funds and financial vehicles, Housing, Rental and leasing services, Social assistance, Gambling and recreation industries, Other services, except government, Federal government enterprises.	Farms, Forestry and fishing, Oil and gas extraction, Mining, Utilities, Construction, Wood products, Fabricated metal, Machinery, Computer and electronics, Electrical equipment, Transportation equipment, Miscellaneous manufacturing, Food and beverage, Textile product mills, Apparel and leather products, Printing and support activities, Petroleum and coal products, Chemical products, Plastics and rubber products, Motor vehicle dealers, Air transportation, Pipeline transportation, Publishing industries, Motion picture and sound recording, Broadcasting and telecommunications, Federal Reserve banks, Other real estate, Legal services, Computer systems design, Miscellaneous technical services, Management of companies, Administrative and support services, Waste management, Educational services, Ambulatory health care services, Hospitals, Nursing and residential care facilities, Performing arts, Accommodation, Food services and drinking places, Federal general government (defense), Federal general government (nondefense).
Short memory	Mining, Support activities for mining, Mineral products, Primary metals, Motor vehicles, Furniture, Paper products, Wholesale trade, Food and beverage stores, General merchandise stores, Other retail, Rail transportation, Water transportation, Truck transportation, Ground transportation, Other transportation activities, Warehousing and storage, Securities and commodity contracts, Insurance carriers, Funds and financial vehicles, Housing, Rental and leasing services, Hospitals, Gambling and recreation industries, Other services, except government, Federal government enterprises.	Farms, Forestrycand fishing, Oil and gas extraction, Utilities, Construction, Wood products, Fabricated metal products, Machinery, Computer and electronics, Electrical equipment, Transportation equipment, Miscellaneous manufacturing, Food and beverage, Textile product mills, Apparel and leather products, Printing and support activities, Petroleum and coal products, Chemical products, Plastics and rubber products, Motor vehicle dealers, Air transportation, Pipeline transportation, Publishing industries, Motion picture and sound recording, Broadcasting and telecommunications, Internet publishing and information services, Federal Reserve banks, Other real estate, Legal services, Computer systems design, Miscellaneous technical services, Management of companies, Administrative and support services, Waste management, Educational services, Ambulatory health care services, Nursing and residential care facilities, Social assistance, Performing arts, Accommodation, Food services and drinking places, Federal general government (defense), Federal general government (nondefense).

According to Table 1, the only industries whose upward versus downward classifications do not coincide in the two learning methods are: *Internet publishing and information services*, *Social assistance*, *Mining*, *Hospitals*. To explore further these discrepancies, Table 2 provides a compact view of the differences of both estimations at the aggregate level, based on summary statistics over the 69 sectors. The correlations between the resulting estimates at the bottom of Table 2 supports the consistency between the two learning methods (cor = 0.920, and cor = 0.944, for the long and short memory, respectively), as well as the estimation of the material input intensity ($\phi = 0.431$, and $\phi = 0.432$, for the long and short memory, respectively).

While these two estimations are compatible in terms of A and ϕ (as also supported by Figure 1 and Figure 2), seemingly larger divergences appear in terms of $v_i^{(0)}$ and $v_i^{(*)}$. Specifically, all the values of $v_i^{(0)}$ and $v_i^{(*)}$ estimated under the long memory assumption are larger than the ones estimated under the short memory assumption, suggesting a non-negligible impact of the learning assumption on the least square estimation of $v_i^{(0)}$ and $v_i^{(*)}$. Although the correlation between $v_i^{(0)}$ and $v_i^{(*)}$ is negative for both learning methods, it is stronger when the long memory assumption is adopted (cor = -0.526) than for the case of short memory (cor = -0.382). Hence, sectors with increasing returns to scale have a tendency to underestimate their parameters, and vice-versa for those with decreasing returns to scale.

Table 2: The estimated returns to scale parameters $v_i^{(0)}$ and $v_i^{(*)}$.

	$v_i^{(0)}$			$v_i^{(*)}$				
	Min	Median	Max	Min	Median	Max	corr	ϕ
Long memory	0.869	1.332	1.915	0.463	1.285	1.524	-0.526	0.431
Short memory	0.645	1.149	1.429	0.463	1.091	1.364	-0.382	0.432
corr		0.920			0.944			

Beyond providing an empirical grounding for our theoretical results, this estimation approach attests that the endogenous learning in input-output economies can be tested in practice, using state-of-the-art statistical methodologies, once the learning equations are consistently embedded in a multi-sector general equilibrium model.

7 Conclusions

In this work, a multi-sector dynamic general equilibrium model is embedded into the framework information collection and processing by economic agents (El-Gamal and Sundaram, 1993; Cogley and Sargent, 2008), with a view to address the problem of endogenous learning of profit-relevant parameters by profit-maximizing firms. The proposed model allowed addressing the fundamental economic question concerning the ability of firms to discover their own production processes.

To do so, we considered a collection of representative firms unaware of the values of certain profit-relevant parameters, that operate in a non-deterministic production environment. We focused on the specific case of the returns to scale parameter, in line with a long stream of contributions (Starrett, 1977; Golany and Yu, 1997; Basu and Fernald, 1997; Banker et al., 2004; Ackerberg et al., 2015; Gong and Hu, 2016). Contextually, the proposed learning mechanism relied on an incomplete but perfect information setting, where firms dynamically observe the realized production outcomes after taking input decisions, while lacking a complete knowledge of their own returns to scale.

Our results shed light on how idiosyncratic shocks translate into learning dynamics of the returns to scale and input-output elasticity structure, providing insights on some fundamental aspects.

- We provided a closed-form characterization of the MAP estimator for the firms' update of their beliefs about the returns to scale. Contextually, we showed that all the relevant information to compute this MAP estimator is encoded in the input decisions.
- We characterized closed-form conditions under which firms are able to learn the true returns to scale (namely, the belief sequence converges to the true parameter), and we showed that these conditions are exclusively related to the the manner in which input decisions are taken.
- We showed that, under the long-memory approach, substantial and enduring economic growth, coupled with a consequent rise in the series of input quantities, enhances the dynamic learning of the true returns to scale. This establishes a theoretical relationship between learning and economic growth, which has been already conjectured since the seminal work of Arrow (1962).
- Under the specific case of constant returns to scale, the mismatch between the true (unknown) returns to scale and those predicted by firms has a critical effect on input decisions. In this vein, we established a price condition which is consistent not only with the correct knowledge of the returns to scale $v_i^{(*)}$, but also with any $v_i(t)$, as long as $\{v_i(t)\}_t$ converges. Hence, for input decisions not to be affected by firms inferential error, the limit prices must reflect the centrality of sectors in the multi-sector network.
- We proposed an estimation approach that allows the empirical testing of the endogenous learning in multi-sector economies, and uncovered empirical figures about upward and downward learning sectors.

Overall, this contribution establishes an integrated modelling approach to address the problem of learning by economic agents, from within the general equilibrium framework. This paves the way to further extensions to address the learning of different profit-relevant parameters, based on the proposed approach. In this regard, a first line of further research that is worth mentioning is to extend the uncertainty about the returns to scale to the uncertainty about all input-output elasticities. Secondly, future research can focus on applying of the proposed MAP estimator in the Bayesian general equilibrium of Toda (2015), where non-optimizing agents respond to prices by setting a prior distribution on their demand. Likewise, the impact of public policy on the speed of convergence remains to be studied. Finally, the inclusion of durable capital inputs (which implies inter-temporal decisions by firms) might have an impact on the learning dynamics and deserves to be part of a future analysis.

Appendix A: Proofs of propositions

Lemma 1

Proof. Consider the problem (1) of a representative household maximizing her lifetime utility, subject to the inter-temporal budget constraint. The first order conditions of the corresponding Lagrangian function imply

$$\frac{\kappa_i}{\kappa_j}\frac{c_j(t)}{c_i(t)} = \frac{p_i(t)}{p_j(t)}, \tag{37}$$

for each pairs of sectors (i, j). After replacing (37) into the budget constraint, the demand curve for the i-th sector is obtained (7).

Next, by substituting the optimal $c_i(t)$ back into the inter-temporal budget constraint, the consumption-saving problem (1) can be defined in terms of the variables $a(t)$:

$$\max_{a(t)\,:\,t=1\ldots T-1} \quad \sum_{t=1}^{T-1} \rho^t \sum_{i\in\mathcal{N}} \kappa_i \log\left(\frac{\kappa_i}{p_i(t)}\left(w(t) - \bar{p}(a(t+1) - a(t)) + \vartheta(t)a(t)\right)\right), \tag{38}$$

when wage $w(t+1)$ and dividend $\vartheta(t+1)$ are known by the household at time t, and

$$\max_{a(t)\,:\,t=1\ldots T-1} \quad \mathbb{E}\left[\sum_{t=1}^{T-1} \rho^t \sum_{i\in\mathcal{N}} \kappa_i \log\left(\frac{\kappa_i}{p_i(t)}\left(w(t) - \bar{p}(a(t+1) - a(t)) + \vartheta(t)a(t)\right)\right)\right], \tag{39}$$

when wage $w(t+1)$ and dividend $\vartheta(t+1)$ are unknown by the household at time t. For the case of (38), the first-order conditions with respect to $a(t+1)$ yield (for each $1 \le t \le T-2$)

$$\frac{\bar{p}}{w(t) - \bar{p}(a(t+1) - a(t)) + \vartheta(t)a(t)} = \frac{\rho(\bar{p} + \vartheta(t))}{w(t+1) - \bar{p}(a(t+2) - a(t+1)) + \vartheta(t+1)a(t+1)}.$$

For the case of (39), the first-order conditions with respect to $a(t+1)$ yield (for each $1 \le t \le T-2$)

$$\frac{\bar{p}}{w(t) - \bar{p}(a(t+1) - a(t)) + \vartheta(t)a(t)} = \mathbb{E}\left[\frac{\rho(\bar{p} + \vartheta(t))}{w(t+1) - \bar{p}(a(t+2) - a(t+1)) + \vartheta(t+1)a(t+1)}\right].$$

□

Lemma 2

Proof. Consider the problem of the i-th representative firm independently maximizing its profit (4) at the end of each production period with respect to the input decisions $y_{i,j}(t)$, subject to the non-negativity constraints $y_{i,j}(t) \ge 0$, for $j = 1 \ldots n$. We distinguish two cases.

First case: $v_i > 0$. Since $\mathbb{E}_{\boldsymbol{\eta}}[r_i(t) - e_i(t)|v_i]$ is concave in $\{l_i(t), y_{i1}(t), \ldots, y_{in}(t)\}$, for each $i \in \mathcal{N}_+$, the equilibrium point is characterized by the corresponding first-order conditions:

$$l_i(t) = \frac{(1-\phi)\mathbb{E}_{\boldsymbol{\eta}}[r_i(t)]}{w(t+1)} \qquad \text{and} \qquad y_{i,j}(t) = \frac{\phi\alpha_{i,j}v_i\mathbb{E}_{\boldsymbol{\eta}}[r_i(t)]}{p_j(t)}, \tag{40}$$

when $0 < \phi < 1$. Note that when $\alpha_{i,j} = 0$, the optimal decision is in the boundary, as $y_{i,j}(t) = 0$. By contrast, as long as $\alpha_{i,j} > 0$, the non-negativity of labor and material input does not need to be explicitly included, as any $l_i(t)$ and $y_{i,j}(t)$ solving (40) are non-negative. Replacing $\mathbb{E}_{\boldsymbol{\eta}}[r_i(t)]$ with $q_i p_i(t+1)x_i(t)$ yields (10) and (11).

Second case: $v_i = 0$. In this case $\mathbb{E}_{\boldsymbol{\eta}}[r_i(t) - e_i(t)|v_i]$ is decreasing in $y_{i,1}(t), \ldots y_{i,n}(t)$, so that the optimal solution of the profit maximizing representative firm is $y_{i,j} = 0$, for all $i \in \mathcal{N}_0$, $j \in \mathcal{N}$. Hence, the first-order conditions with respect to $l_i(t)$ yields:[16]

$$p_i(t+1)(1-\phi)\frac{1}{l_i(t)^\phi} = w(t+1),$$

which implies (12). □

Proposition 1

Proof. We drop the index i and the time period t from $v_i(t)$, as this proof is valid for all firms and all periods. Based on Assumption 3, $\pi_{0,i}$ is the density function of a zero-truncated Gaussian random variable with parameters $v_i^{(0)}$ and τ_i. For the case $y_{i,j}(\ell)^{\phi\alpha_{i,j}} > 0$ for all $\ell = 1, \ldots, t$ and $j \in \mathcal{N}$, we have $z_i(\ell) > -\infty$, for all $\ell = 1, \ldots, t$, so that we can write

$$\mathscr{L}(v;\, \mathcal{I}_i(t))\pi_{0,i}(v) \quad \propto \begin{cases} \exp\left(-\frac{1}{2}\theta_i(v, \mathcal{I}_i(t))\right) & \text{if } v \geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

where θ_i is defined as

$$\theta_i(v, \mathcal{I}_i(t)) \;=\; \begin{cases} \displaystyle\sum_{\ell=1}^{t} \frac{1}{\sigma_i^2}\left(s_i(\ell) - v z_i(t)\right)^2 + \frac{1}{\tau_i^2}\left(v - v_i^{(0)}\right)^2 & \text{if } t > 1 \\ \displaystyle\frac{1}{\tau_i^2}\left(v - v_i^{(0)}\right)^2, & \text{if } t = 1. \end{cases}$$

To maximize the posterior distribution for each period t, it is sufficient to solve

$$\max_v \quad \exp\left(-\theta_i(v, \mathcal{I}_i(t))\right), \qquad \text{subject to } v \geq 0,$$

as $\int \mathscr{L}(v;\, \mathcal{I}_i(t))\pi_{0,i}(v)\; dv$ is constant with respect to v. By the Karush-Kuhn-Tucker conditions:

$$\frac{\partial}{\partial v}\left[\exp\left(-\frac{1}{2}\theta_i(v, \mathcal{I}_i(t))\right) - \zeta v\right] \;=\; 0 \quad \text{and} \quad \zeta v = 0,$$

so that

$$\exp\left(-\frac{1}{2}\theta_i(v, \mathcal{I}_i(t))\right)\left(\frac{\partial}{\partial v}\theta_i(v, \mathcal{I}_i(t))\right) = \zeta.$$

Due to the complementarity either $\zeta = 0$ or $v = 0$. Therefore, if $v > 0$, then $\zeta = 0$ and $\frac{\partial}{\partial v}\theta_i(v, \mathcal{I}_i(t)) = 0$. This implies that for any $i \in \mathcal{N}$, we have $v_i(t+1)$ is the maximum between zero and the solution of

$$\begin{cases} \displaystyle\sum_{\ell=1}^{t} \frac{1}{\sigma_i^2} z_i(t)\Big(s_i(\ell) - v_i(t+1) z_i(t)\Big) = \frac{1}{\tau_i^2}\left(v_i(t+1) - v_i^{(0)}\right) & \text{if } t > 1 \\ v_i(t+1) = v_i^{(0)} & \text{if } t = 1. \end{cases}$$

[16]Since $\lim_{z\to 0^+} z^z = 1$, we adopt the convention $0^0 = 1$.

Therefore, we obtain (17), for the case $y_{i,j}(\ell)^{\phi\alpha_{i,j}} > 0$ for all $\ell = 1, \ldots, t$ and $j \in \mathcal{N}$. Then, by continuous extension, if $y_{ij}(l) \to 0$ for some i, j, l, $z_i(l) \to -\infty$. Therefore

$$v_i(t+1) = \lim_{y_{ij}(l)\to 0} \left(\frac{v_i^{(0)} + \gamma_i \sum_{\ell=1}^t z_i(\ell) s_i(\ell)}{\left(1 + \gamma_i \sum_{\ell=1}^t z_i(\ell)^2\right)} \right)^+ = \lim_{y_{ij}(l)\to 0} \frac{v_i^{(*)} \gamma_i \sum_{\ell=1}^t z_i(\ell) z_i(\ell)}{\gamma_i \sum_{\ell=1}^t z_i(\ell)^2} = v_i^{(*)}.$$

□

Proposition 2

Proof. Let $\mathcal{Y}(t) = \{Y(\ell)\}_{\ell=0}^t$, and for a given $i \in \mathcal{N}$ define:

$$b_i(t) = -\left| \frac{\overline{v}_i(\mathcal{Y}(t))}{\overline{\varphi}_i(\mathcal{Y}(t))} \right| \quad \text{and} \quad U_i(t) = \overline{v}_i(\mathcal{Y}(t)) + \overline{\varphi}_i(\mathcal{Y}(t))\varepsilon,$$

where $\varepsilon \sim N(0,1)$. We distinguish two cases.

- Case 1. If $\sum_{\ell=1}^t z_i(\ell) > 0$, we have,

$$\begin{aligned} \mathbb{E}[\max(0, U_i(t))] &= \mathbb{E}[(\overline{v}_i(\mathcal{Y}(t)) + \overline{\varphi}_i(\mathcal{Y}(t))\varepsilon)^+] \\ &= \overline{v}_i(\mathcal{Y}(t)) \int_{b_i(t)}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\varepsilon^2/2} d\varepsilon \; + \; \overline{\varphi}_i(\mathcal{Y}(t)) \int_{b_i(t)}^{\infty} \varepsilon \frac{e^{-\varepsilon^2/2}}{\sqrt{2\pi}} d\varepsilon \qquad (41) \\ &= \overline{v}_i(\mathcal{Y}(t))(1 - F(b_i(t))) \; + \; \overline{\varphi}_i(\mathcal{Y}(t)) \int_{b_i(t)}^{\infty} \varepsilon \frac{e^{-\varepsilon^2/2}}{\sqrt{2\pi}} d\varepsilon. \end{aligned}$$

  To compute the latter integral, we define

$$B(b, m) = \int_b^{\infty} \varepsilon^m \frac{1}{\sqrt{2\pi}} e^{-\varepsilon^2/2} d\varepsilon$$

  and note that using the integration by parts

$$\begin{aligned} B(b, m-2) &= \int_b^{\infty} \varepsilon^{m-2} \frac{1}{\sqrt{2\pi}} e^{-\varepsilon^2/2} d\varepsilon \\ &= \frac{1}{m-1} \left(\frac{1}{\sqrt{2\pi}} \lim_{x\to\infty} \left[x^{m-1} e^{-x^2/2} \right] - b^{m-1} f(b) + B(b, m) \right) \\ &= \frac{1}{m-1} \left(B(b, m) - b^{m-1} f(b) \right) \end{aligned}$$

  where the last equality comes from the fact that

$$0 \leq \lim_{x\to\infty} \left[x^{m-1} e^{-x^2/2} \right] = \lim_{x\to\infty} \left[x e^{-\frac{x^2}{2(m-1)}} \right]^{m-1} \leq \lim_{x\to\infty} \left[e^{-\frac{x^2}{2(m-1)} + x} \right]^{m-1} = 0.$$

  We obtain the recurrence relation $B(b, m) = (m-1)B(b, m-2) + b^{m-1} f(b)$, so that the first partial moment is $B(b, 1) = f(b)$. Substituting it back into (41), we obtain the expected $v_i(t+1)$ in (25).

$$\begin{aligned} \mathbb{E}[v_i(t+1)] &= \overline{v}_i(\mathcal{Y}(t))(1 - F(b_i(t))) \; + \; \overline{\varphi}_i(\mathcal{Y}(t)) f(b_i(t)) \\ &= \frac{\left(v_i^{(0)} + \gamma_i v_i^{(*)} \tilde{z}_i^{(2)}(t)\right)(1 - F(b_i(t))) + \gamma_i \sigma_i \tilde{z}_i^{(1)}(t) f(b_i(t))}{1 + \gamma_i \tilde{z}_i^{(2)}(t)}. \end{aligned}$$

Applying the same procedure for the second order moment, and using the recurrence relation $B(b,m) = (m-1)B(b,m-2) + b^{m-1}f(b)$, with the initial conditions

$$\begin{cases} B(b,0) &= \int_b^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\varepsilon^2/2} d\varepsilon &= 1 - F(b), \\ B(b,1) &= 0B(b,-1) + f(b) &= f(b), \end{cases}$$

we have $B(b,2) = B(b,0) + bf(b)$, so that

$$\begin{aligned} \mathbb{E}[((U_i(t))^+)^2] &= \mathbb{E}[((\overline{v}_i(\mathcal{Y}(t)) + \overline{\varphi}_i(\mathcal{Y}(t))\varepsilon)^+)^2] \\ &= \left[\overline{v}_i(\mathcal{Y}(t))^2 + \overline{\varphi}_i(\mathcal{Y}(t))^2\right][1 - F(b_i(t))] + \overline{v}_i(\mathcal{Y}(t))\overline{\varphi}_i(\mathcal{Y}(t))f(b_i(t)). \end{aligned}$$

- Case 2. If $\sum_{\ell=1}^{t} z_i(\ell) < 0$, we have,

$$\begin{aligned} \mathbb{E}[U_i(t)^+] &= \int_{-\infty}^{b_i(t)} (\overline{v}_i(\mathcal{Y}(t)) + \overline{\varphi}_i(\mathcal{Y}(t))\varepsilon) \frac{1}{\sqrt{2\pi}} e^{-\varepsilon^2/2} d\varepsilon \\ &= \overline{v}_i(\mathcal{Y}(t))F(b_i(t)) \;+\; \overline{\varphi}_i(\mathcal{Y}(t)) \int_{-\infty}^{b_i(t)} \varepsilon \frac{1}{\sqrt{2\pi}} e^{-\varepsilon^2/2} d\varepsilon. \end{aligned}$$

Since $\int_{-\infty}^{b_i} \varepsilon \frac{1}{\sqrt{2\pi}} e^{-\varepsilon^2/2} d\varepsilon = -\int_{b_i}^{+\infty} \varepsilon \frac{1}{\sqrt{2\pi}} e^{-\varepsilon^2/2} d\varepsilon = -f(b_i)$. Hence,

$$\begin{aligned} \mathbb{E}[v_i(t+1)] &= \overline{v}_i(\mathcal{Y}(t))F(b_i(t)) \;-\; \overline{\varphi}_i(\mathcal{Y}(t))f(b_i(t)) \\ &= \frac{\left(v_i^{(0)} + \gamma_i v_i^{(*)} \tilde{z}_i^{(2)}(t)\right) F\left(b_i(t)\right) - \gamma_i \sigma_i \tilde{z}_i^{(1)}(t) f\left(b_i(t)\right)}{1 + \gamma_i \tilde{z}_i^{(2)}(t)}. \end{aligned}$$

To compute the latter integral, we define

$$\overline{B}(b,m) = \int_{-\infty}^{b} \varepsilon^m \frac{1}{\sqrt{2\pi}} e^{-\varepsilon^2/2} d\varepsilon.$$

By the same arguments, we obtain that $\overline{B}(b,m) = (m-1)\overline{B}(b,m-2) - b^{m-1}f(b)$. Therefore

$$\mathbb{E}[((U_i(t))^+)^2] = \left[\overline{v}_i(\mathcal{Y}(t))^2 + \overline{\varphi}_i(\mathcal{Y}(t))^2\right] F(b_i(t)) - \overline{v}_i(\mathcal{Y}(t))\overline{\varphi}_i(\mathcal{Y}(t))f(b_i(t)).$$

□

Proposition 3

Proof. Using (17) and (18), we consider the probability density function of $v_i(t+1)$, conditioned on the input decisions

$$\mathbb{P}(v_i(t+1) \mid \mathcal{Y}(t)) = \begin{cases} \dfrac{\exp\left(-\dfrac{(v_i(t) - \overline{v}_i(\mathcal{Y}(t)))^2}{2|\overline{\varphi}_i(\mathcal{Y}(t))|^2}\right)}{|\overline{\varphi}_i(\mathcal{Y}(t))|\sqrt{2\pi}}, & \text{if } v_i(t+1) > 0 \\ F\left(-\overline{v}_i(\mathcal{Y}(t))/\overline{\varphi}_i(\mathcal{Y}(t))\right), & \text{if } v_i(t+1) = 0 \\ 0, & \text{otherwise.} \end{cases}$$

Note that for $v_i(t+1) > 0$, if $\overline{v}_i(\mathcal{Y}(t))) > 0$, then $\mathbb{P}(v_i(t+1) \mid \mathcal{Y}(t))$ is maximized when $v_i(t+1) = \overline{v}_i(\mathcal{Y}(t)))$, with $\mathbb{P}(\overline{v}_i(\mathcal{Y}(t))) \mid \mathcal{Y}(t)) = (|\overline{\varphi}_i(\mathcal{Y}(t))|\sqrt{2\pi})^{-1}$. Likewise, $\mathbb{P}(\, 0 \mid \mathcal{Y}(t)) = F\left(-\overline{v}_i(\mathcal{Y}(t))/\overline{\varphi}_i(\mathcal{Y}(t))\right)$. Therefore,

$$\max_{v_i(t+1)} \mathbb{P}(v_i(t+1) \mid \mathcal{Y}(t)) = \begin{cases} \overline{v}_i(\mathcal{Y}(t)) & \text{if } \dfrac{1}{|\overline{\varphi}_i(\mathcal{Y}(t))|\sqrt{2\pi}} \geq F\left(-\overline{v}_i(\mathcal{Y}(t))/\overline{\varphi}_i(\mathcal{Y}(t))\right), \\ 0, & \text{otherwise.} \end{cases}$$

□

Proposition 4

Proof. We distinguish the following cases: Recall the following sequence (defined in (18))

$$\overline{v}_i(t) = \frac{v_i^{(0)} + \gamma_i v_i^{(*)} \tilde{z}_i^{(2)}(t)}{\gamma_i \sigma_i \tilde{z}_i^{(1)}(t)}.$$

(1) If $\lim_{t\to+\infty} |\tilde{z}_i^{(1)}(t)| = \lim_{t\to+\infty} \tilde{z}_i^{(2)}(t) = +\infty$ and $\lim_{t\to+\infty} \overline{v}_i(t) = L < +\infty$, then $\overline{v}_i(t) = L + o(t)$ with $\lim_{t\to+\infty} o(t) = 0$. Then,

$$\begin{aligned}\lim_{t\to+\infty} \mathbb{E}[v_i(t)] &= \lim_{t\to+\infty} \frac{\left(v_i^{(0)} + \gamma_i v_i^{(*)} \tilde{z}_i^{(2)}(t)\right)(1 - F(-L - o(t)) + \sigma_i\gamma_i |\tilde{z}_i^{(1)}(t)| f(-L - o(t))}{1 + \gamma_i \tilde{z}_i^{(2)}(t)} \\ &= v_i^{(*)}(1 - F(-L)) + \sigma_i f(-L) \lim_{t\to+\infty} \frac{|\tilde{z}_i^{(2)}(t)|}{\tilde{z}_i^{(1)}(t)}.\end{aligned}$$

(2) If $\lim_{t\to+\infty} \tilde{z}_i^{(1)}(t) = L_1 \leq +\infty$ and $\lim_{t\to+\infty} \tilde{z}_i^{(2)}(t) = L_2 \leq +\infty$, let us consider the following function

$$\Upsilon(x, y) = \frac{\left(v_i^{(0)} + \gamma_i v_i^{(*)} y\right) G_i(x, y) + \sigma_i \gamma_i x g_i(x, y)}{1 + \gamma_i y}.$$

Since Υ is continuous in $\mathbb{R}_+^* \times \mathbb{R}_+$, then we need to distinguish the two following cases:

(2.1) If $L_1 \neq 0$, then by continuity of Υ, we obtain that

$$\lim_{t\to+\infty} \mathbb{E}[v_i(t)] = \frac{\left(v_i^{(0)} + \gamma_i v_i^{(*)} L_2\right) G_i(L_1, L_2) + \sigma_i \gamma_i L_1 g_i(L_1, L_2)}{1 + \gamma_i L_2}.$$

(2.2) If $L_1 = 0$, then $g_i(\tilde{z}_i^{(1)}(t), \tilde{z}_i^{(2)}(t)) = o_1(t)$ and $G_i(\tilde{z}_i^{(1)}(t), \tilde{z}_i^{(2)}(t)) = 1 + o_2(t)$ with $\lim_{t\to+\infty} o_1(t) = \lim_{t\to+\infty} o_2(t) = 0$. Then,

$$\mathbb{E}[v_i(t)] = \frac{\left(v_i^{(0)} + \gamma_i v_i^{(*)} \tilde{z}_i^{(1)}(t)\right)(1 + o_2(t)) + \sigma_i \gamma_i |\tilde{z}_i^{(1)}(t)| o_1(t)}{1 + \gamma_i \tilde{z}_i^{(2)}(t)}.$$

Therefore, we obtain

$$\begin{aligned}\lim_{t\to+\infty} \mathbb{E}[v_i(t)] &= \lim_{t\to+\infty} \frac{\left(v_i^{(0)} + \gamma_i v_i^{(*)} \tilde{z}_i^{(2)}(t)\right)(1 + o_2(t)) + \sigma_i \gamma_i |\tilde{z}_i^{(1)}(t)| o_1(t)}{1 + \gamma_i \tilde{z}_i^{(2)}(t)} \\ &= \lim_{t\to+\infty} \frac{\left(v_i^{(0)} + \gamma_i v_i^{(*)} \tilde{z}_i^{(2)}(t)\right)(1 + o_2(t))}{1 + \gamma_i \tilde{z}_i^{(2)}(t)} \\ &= \frac{v_i^{(0)} + \gamma_i v_i^{(*)} L_2}{1 + \gamma_i L_2}.\end{aligned}$$

(3) If $\lim_{t\to+\infty} \tilde{z}_i^{(1)}(t) = L_1$ and $\lim_{t\to+\infty} \tilde{z}_i^{(2)}(t) = \infty$, then $g_i(\tilde{z}_i^{(1)}(t), \tilde{z}_i^{(2)}(t)) = o_1(t)$ and $G_i(\tilde{z}_i^{(1)}(t), \tilde{z}_i^{(2)}(t)) = 1 + o_2(t)$ with $\lim_{t\to+\infty} o_1(t) = \lim_{t\to+\infty} o_2(t) = 0$. Then,

$$\mathbb{E}[v_i(t)] = \frac{\left(v_i^{(0)} + \gamma_i v_i^{(*)} \tilde{z}_i^{(2)}(t)\right)(1 + o_2(t)) + \sigma_i\gamma_i|\tilde{z}_i^{(1)}(t)|o_1(t)}{1 + \gamma_i \tilde{z}_i^{(1)}(t)}.$$

Therefore,

$$\begin{aligned}
\lim_{t\to+\infty} \mathbb{E}[v_i(t)] &= \lim_{t\to+\infty} \frac{\left(v_i^{(0)} + \gamma_i v_i^{(*)} \tilde{z}_i^{(2)}(t)\right)(1 + o_2(t)) + \sigma_i\gamma_i|\tilde{z}_i^{(1)}(t)|o_{(1)}(t)}{1 + \gamma_i \tilde{z}_i^{(2)}(t)} \\
&= \lim_{t\to+\infty} \frac{\left(v_i^{(0)} + \gamma_i v_i^{(*)} \tilde{z}_i^{(2)}(t)\right)}{1 + \gamma_i \tilde{z}_i^{(2)}(t)} \\
&= v_i^{(*)}.
\end{aligned}$$

(4) If $\lim_{t\to+\infty} \tilde{z}_i^{(1)}(t) = \infty$ and $\lim_{t\to+\infty} \tilde{z}_i^{(2)}(t) = L_2$, then $g_i(\tilde{z}_i^{(1)}(t), \tilde{z}_i^{(2)}(t)) = \frac{1}{\sqrt{2\pi}} + o_1(t)$ and $G_i(\tilde{z}_i^{(1)}(t), \tilde{z}_i^{(2)}(t)) = \frac{1}{2} + o_2(t)$ with $\lim_{t\to+\infty} o_1(t) = \lim_{t\to+\infty} o_2(t) = 0$. Then,

$$\mathbb{E}[v_i(t)] = \frac{\left(v_i^{(0)} + \gamma_i v_i^{(*)} \tilde{z}_i^{(2)}(t)\right)(\frac{1}{2} + o_2(t)) + \sigma_i\gamma_i|\tilde{z}_i^{(1)}(t)|\left(\frac{1}{\sqrt{2\pi}} + o_1(t)\right)}{1 + \gamma_i \tilde{z}_i^{(2)}(t)}.$$

Therefore,

$$\begin{aligned}
\lim_{t\to+\infty} \mathbb{E}[v_i(t)] &= \frac{\left(v_i^{(0)} + \gamma_i v_i^{(*)} \tilde{z}_i^{(2)}(t)\right)(\frac{1}{2} + o_2(t)) + \sigma_i\gamma_i|\tilde{z}_i^{(1)}(t)|\left(\frac{1}{\sqrt{2\pi}} + o_1(t)\right)}{1 + \gamma_i \tilde{z}_i^{(2)}(t)} \\
&= \frac{\sigma_i\gamma_i|\tilde{z}_i^{(1)}(t)|\frac{1}{\sqrt{2\pi}}}{1 + \gamma_i \tilde{z}_i^{(2)}(t)} \\
&= \lim_{t\to+\infty} |\tilde{z}_i^{(1)}(t)| \\
&= +\infty.
\end{aligned}$$

(5) If $\lim_{t\to+\infty} |\tilde{z}_i^{(1)}(t)| = \lim_{t\to+\infty} \tilde{z}_i^{(2)}(t) = \lim_{t\to+\infty} \frac{v_i^{(0)} + \gamma_i v_i^{(*)} \tilde{z}_i^{(2)}(t)}{\gamma_i \sigma_i \tilde{z}_i^{(1)}(t)} = +\infty$, then $g_i(\tilde{z}_i^{(1)}(t), \tilde{z}_i^{(2)}(t)) = o_1(t)$ and $G_i(\tilde{z}_i^{(1)}(t), \tilde{z}_i^{(2)}(t)) = 1 + o_2(t)$ with $\lim_{t\to+\infty} o_1(t) = \lim_{t\to+\infty} o_2(t) = 0$. Hence,

$$\mathbb{E}[v_i(t)] = \frac{\left(v_i^{(0)} + \gamma_i v_i^{(*)} \tilde{z}_i^{(2)}(t)\right)(1 + o_2(t)) + \sigma_i\gamma_i|\tilde{z}_i^{(1)}(t)|o_1(t)}{1 + \gamma_i \tilde{z}_i^{(2)}(t)}.$$

Therefore,

$$\begin{aligned}
\lim_{t\to+\infty} \mathbb{E}[v_i(t)] &= \lim_{t\to+\infty} \frac{\left(v_i^{(0)} + \gamma_i v_i^{(*)} \tilde{z}_i^{(2)}(t)\right)(1 + o_2(t)) + \sigma_i\gamma_i|\tilde{z}_i^{(1)}(t)|o_1(t)}{1 + \gamma_i \tilde{z}_i^{(2)}(t)} \\
&= v_i^{(*)} + \lim_{t\to+\infty} \frac{|\tilde{z}_i^{(1)}(t)|o_1(t)}{\tilde{z}_i^{(2)}(t)}
\end{aligned}$$

is undetermined.

□

Proposition 5

Proof. We invoke Proposition 3 to characterize the mode of the MAP estimator:

$$\mathbb{M}[v_i(t) \mid \mathcal{Y}(t)] = \underset{v_i(t+1)}{\operatorname{argmax}} \ \mathbb{P}(v_i(t+1) \mid \mathcal{Y}(t)) = \begin{cases} \overline{v}_i(\mathcal{Y}(t)) & \text{if } \dfrac{1}{|\overline{\varphi}_i(\mathcal{Y}(t))|\sqrt{2\pi}} \geq F\left(-\left|\dfrac{\overline{v}_i(\mathcal{Y}(t))}{\overline{\varphi}_i(\mathcal{Y}(t))}\right|\right), \\ 0 & \text{otherwise.} \end{cases}$$

We construct the ordered set $\overline{\Psi}_i$ (as defined in (21)) as the collection of time periods for which $\mathbb{M}[v_i(t) \mid \mathcal{Y}(t)] > 0$. This allows constructing a sub-sequence by indexing the elements of $\overline{\Psi}_i$ by $\varrho(t)$, for $t \in \mathbb{N}$. Hence,

$$\{ \ \mathbb{M}[v_i(t) \mid \mathcal{Y}(t)] \ \}_{t \in \overline{\Psi}_i} \equiv \{ \ \overline{v}_i(\varrho(t)) \ \}_{t \geq 0}$$

Focusing on the sub-sequence $\{\overline{v}_i(\varrho(t))\}_{t \geq 0}$, we distinguish the following cases:

(1) If $\overline{\Psi}_i$ is a finite set, then there is a $t_0 \in \mathbb{N}$ such that for each $t \geq t_0$, we have $\mathbb{M}[v_i(t) \mid \mathcal{Y}(t)] = 0$ and therefore $\{\mathbb{M}[v_i(t) \mid \mathcal{Y}(t)]\}_t$ converges to 0.

(2) Assume that $\overline{\Psi}_i$ is an infinite set. Then, without loss of generality, let $\overline{\Psi}_i \equiv \mathbb{N}$ (in other words, if $\overline{\Psi}_i \not\equiv \mathbb{N}$, then there exists a bijection $\varphi : \mathbb{N} \to \varphi(\mathbb{N}) = \overline{\Psi}_i$).

(2.1) If $\lim_{t\to\infty} \tilde{z}_i^{(2)}(t) = +\infty$, then $\mathbb{M}[v_i(t) \mid \mathcal{Y}(t)] = \dfrac{v_i^{(0)} + \gamma_i v_i^{(*)} \tilde{z}_i^{(2)}(t)}{1 + \gamma_i \tilde{z}_i^{(2)}(t)}$. As a consequence, we obtain

$$\begin{aligned} \lim_{t\to+\infty} \mathbb{M}[v_i(t) \mid \mathcal{Y}(t)] &= \lim_{t\to+\infty} \frac{v_i^{(0)} + \gamma_i v_i^{(*)} \tilde{z}_i^{(2)}(t)}{1 + \gamma_i \tilde{z}_i^{(2)}(t)} \\ &= \lim_{t\to+\infty} \frac{\gamma_i v_i^{(*)} \tilde{z}_i^{(2)}(t)}{1 + \gamma \tilde{z}_i^{(2)}(t)} \\ &= v_i^{(*)}. \end{aligned}$$

(2.2) If $\lim_{t\to\infty} \tilde{z}_i^{(2)}(t) = L_2 < +\infty$, then there exists a function $o(t)$ such that $\tilde{z}_i^{(2)}(t) = L_2 + o(t)$ with $\lim_{t\to+\infty} o(t) = 0$. Then we obtain that

$$\begin{aligned} \lim_{t\to+\infty} \mathbb{M}[v_i(t) \mid \mathcal{Y}(t)] &= \lim_{t\to+\infty} \frac{v_i^{(0)} + \gamma_i v_i^{(*)}(L_2 + o(t))}{1 + \gamma_i(L_2 + o(t))} \\ &= \frac{v_i^{(0)} + \gamma_i v_i^{(*)} L_2}{1 + \gamma_i L_2}. \end{aligned}$$

□

Corollary 1

Proof. We have that

$$\text{if} \quad \lim_{t\to+\infty} \prod_{j=1}^{n} (y_{i,j}(t))^{\phi \alpha_{i,j}} = +\infty \quad \text{then} \quad \lim_{t\to+\infty} z_i(t) = +\infty.$$

In such case, the input decisions are such that $\tilde{z}_i^{(2)}(t) = \sum_{\ell=1}^{t-1} z_i(\ell)^2$ grows faster than $\tilde{z}_i^{(1)}(t) = \sum_{\ell=1}^{t-1} z_i(\ell)$. Therefore, there exists t_0 such that for all $t > t_0$,

$$\frac{1}{|\overline{\varphi}_i(\mathcal{Y}(t))|} = \left|\frac{1+\gamma_i \tilde{z}_i^{(2)}(t)}{\gamma_i \sigma_i \tilde{z}_i^{(1)}(t)}\right| \geq \sqrt{2\pi} \geq \sqrt{2\pi} F\left(-\left|\frac{v_i^{(0)} + \gamma_i v_i^{(*)} \tilde{z}_i^{(2)}(t)}{\gamma_i \sigma_i \tilde{z}_i^{(1)}(t)}\right|\right)$$

which implies that

$$\left\{t \in \mathbb{N} \ : \ |\overline{\varphi}_i(\mathcal{Y}(t))| \sqrt{2\pi} F\left(-\left|\frac{\overline{v}_i(\mathcal{Y}(t))}{\overline{\varphi}_i(\mathcal{Y}(t))}\right|\right) > 1\right\}$$

is a finite set and that $\overline{\Psi}_i$ is an infinite set. Hence, by Proposition 5,

$$\lim_{t\to+\infty} \mathbb{M}[v_i(t) \mid \mathcal{Y}(t)] = v_i^{(*)}.$$

□

Proposition 6 and Proposition 7

Proof. Proposition 6 and Proposition 7 are obtained by replacing $v_i^{(0)}$ with $v_i(t)$ in Proposition 1, and $\tilde{z}_i^{(1)}(t)$ and $\tilde{z}_i^{(2)}(t)$ with $z_i(t)$ and $z_i(t)^2$, respectively, in Proposition 2. To lighten the exposition, we avoid reporting whole expressions extensively.

□

Proposition 8

Proof. We consider the probability density function of $v_i(t)$, conditioned on the input decisions

$$\mathbb{P}(v_i(t+1) \mid \mathcal{Y}(t)) = \begin{cases} \dfrac{\exp\left(-\dfrac{(v_i(t) - \overline{\overline{v}}_i(\mathcal{Y}(t)))^2}{2|\overline{\overline{\varphi}}_i(\mathcal{Y}(t))|^2}\right)}{|\overline{\overline{\varphi}}_i(\mathcal{Y}(t))|\sqrt{2\pi}}, & \text{if } v_i(t) > 0 \\ F\left(-\overline{\overline{v}}_i(\mathcal{Y}(t))/\overline{\overline{\varphi}}_i(\mathcal{Y}(t))\right), & \text{if } v_i(t) = 0 \\ 0, & \text{otherwise.} \end{cases}$$

Note that for $v_i(t) > 0$, if $\overline{\overline{v}}_i(\mathcal{Y}(t))) > 0$, then $\mathbb{P}(v_i(t) \mid \mathcal{Y}(t))$ is maximized when $v_i(t) = \overline{\overline{v}}_i(\mathcal{Y}(t)))$, with $\mathbb{P}(\overline{\overline{v}}_i(\mathcal{Y}(t))) \mid \mathcal{Y}(t)) = \dfrac{1}{|\overline{\overline{\varphi}}_i(\mathcal{Y}(t))|\sqrt{2\pi}}$. Likewise, $\mathbb{P}(\ 0 \mid \mathcal{Y}(t)) = F\left(-\overline{\overline{v}}_i(\mathcal{Y}(t))/\overline{\overline{\varphi}}_i(\mathcal{Y}(t))\right)$. Therefore,

$$\max_{v_i(t)} \ \mathbb{P}(v_i(t) \mid \mathcal{Y}(t)) = \begin{cases} \overline{\overline{v}}_i(\mathcal{Y}(t)) & \text{if } \dfrac{1}{|\overline{\overline{\varphi}}_i(\mathcal{Y}(t))|\sqrt{2\pi}} \geq F\left(-\overline{\overline{v}}_i(\mathcal{Y}(t))/\overline{\overline{\varphi}}_i(\mathcal{Y}(t))\right), \\ 0 & \text{otherwise.} \end{cases}$$

□

Proposition 9

Proof. Let us prove that the following sequence

$$\mathbb{E}_{\varepsilon_i(t)}[v_i(t+1)] = \frac{\mathbb{E}_{\varepsilon_i(t)}[v_i(t)] + \gamma_i v_i^{(*)} z_i(t)^2}{1+\gamma_i z_i(t)^2} G_i(z_i(t), z_i(t)^2) + \frac{\gamma_i \sigma_i |z_i(t)|}{1+\gamma_i z_i(t)^2} g_i(z_i(t), z_t(t)^2)$$

is convergent. For this, let us consider the following sequence

$$B_i(t) = \frac{v_i(t) + \gamma_i v_i^{(*)} z_i(t)^2}{\gamma_i \sigma_i z_i(t)}.$$

We distinguish two cases:

(1) If $\lim_{t\to+\infty} |z_i(t)| = +\infty$, then $\lim_{t\to+\infty} B_i(t) = +\infty$ and consequently, $g_i(z_i(t), z_i(t)^2) = o_1(t)$ and $G_i(z_i(t), z_i(t)^2) = 1 + o_2(t)$ with $\lim_{t\to+\infty} o_1(t) = \lim_{t\to+\infty} o_2(t) = 0$. Then

$$\mathbb{E}_{\varepsilon_i(t)}[v_i(t+1)] = \frac{\left(\mathbb{E}_{\varepsilon_i(t)}[v_i(t)] + \gamma_i v_i^{(*)} z_i(t)\right)(1 + o_2(t)) + \sigma_i\gamma_i|z_i(t)|o_1(t)}{1 + \gamma_i z_i(t)^2}.$$

Therefore, we obtain that

$$\begin{aligned}\lim_{t\to+\infty} \mathbb{E}_{\varepsilon_i(t)}[v_i(t+1)] &= \lim_{t\to+\infty} \frac{\left(\mathbb{E}_{\varepsilon_i(t)}[v_i(t)] + \gamma_i v_i^{(*)} z_i(t)^2\right)(1 + o_2(t)) + \sigma_i\gamma_i|z_i(t)|o_1(t)}{1 + \gamma_i z_i(t)^2} \\ &= \lim_{t\to+\infty} \frac{\mathbb{E}_{\varepsilon_i(t)}[v_i(t)] + \gamma_i v_i^{(*)} z_i(t)^2}{1 + \gamma_i z_i(t)^2}.\end{aligned}$$

Define the following sequence as follows by

$$\bar{V}_i(t) = \mathbb{E}_{\varepsilon_i(t)}[v_i(t)], \text{ and } \bar{V}_i(t+1) = \frac{\bar{V}_i(t) + \gamma_i v_i^{(*)} z_i(t)^2}{1 + \gamma_i z_i(t)^2}.$$

We have the following implications:

I.1. If $\bar{V}_i(0) \geq v_i^{(*)}$, then the sequence $\{\bar{V}_i(t)\}_{t\geq 0}$ is decreasing and lower bounded by $v_i^{(*)}$.

I.2. If $\bar{V}_i(0) \leq v_i^{(*)}$, then the sequence $\{\bar{V}_i(t)\}_{t\geq 0}$ is increasing and upper bounded by $v_i^{(*)}$.

For the first case, by induction we have $\bar{V}_i(0) \geq v_i^{(*)}$ and $X_1 \geq v_i^{(*)}$. Assume that $\bar{V}_i(t) \geq v_i^{(*)}$. If $\bar{V}_i(t+1) < v_i^{(*)}$, then

$$\frac{\bar{V}_i(t) + \gamma_i v_i^{(*)} z_i(t)^2}{(1 + \gamma_i z_i(t)^2)} < v_i^{(*)},$$

which implies that $\bar{V}_i(t) < v_i^{(*)}$ which is a contradiction. Hence,

$$\begin{aligned}\bar{V}_i(t+1) - \bar{V}_i(t) &= \frac{\bar{V}_i(t) + \gamma_i v_i^{(*)} z_i(t)^2}{(1 + \gamma_i z_i(t)^2)} - \bar{V}_i(t) \\ &= \gamma_i v_i^{(*)} z_i(t)^2 \frac{v_i^{(*)} - \bar{V}_i(t)}{(1 + \gamma_i z_i(t)^2)} \\ &\leq 0.\end{aligned}$$

Then the sequence $\{\bar{V}_i(t)\}_{t\geq 0}$ is decreasing. We can prove the second case by an analogous reasoning. We then conclude that the sequence $\{\bar{V}_i(t)\}_{t\geq 0}$ is convergent.

(2) For the case $\lim_{t\to+\infty} |z_i(t)| = L < +\infty$, let us assume that the sequence $\{\mathbb{E}_{\varepsilon_i(t)}[v_i(t)]\}_t$ is divergent. Then, $\lim_{t\to+\infty} B_i(t) = +\infty$ and consequently, $g_i(z_i(t), z_i(t)^2) = o_1(t)$ and $G_i(z_i(t), z_i(t)^2) = 1 + o_2(t)$ with $\lim_{t\to+\infty} o_1(t) = \lim_{t\to+\infty} o_2(t) = 0$. Therefore, we obtain

$$\mathbb{E}_{\varepsilon_i(t)}[v_i(t+1)] = \frac{\left(\mathbb{E}_{\varepsilon_i(t)}[v_i(t)] + \gamma_i v_i^{(*)} z_i(t)\right)(1 + o_2(t)) + \sigma_i\gamma_i|z_i(t)|o_1(t)}{1 + \gamma_i z_i(t)^2},$$

which implies

$$\begin{aligned}\lim_{t\to+\infty}\mathbb{E}_{\varepsilon_i(t)}[v_i(t+1)] &= \lim_{t\to+\infty}\frac{\left(\mathbb{E}_{\varepsilon_i(t)}[v_i(t)]+\gamma_i v_i^{(*)}z_i(t)^2\right)(1+o_2(t))+\sigma_i\gamma_i|z_i(t)|o_1(t)}{1+\gamma_i z_i(t)^2}\\ &= \lim_{t\to+\infty}\frac{\mathbb{E}_{\varepsilon_i(t)}[v_i(t)]+\gamma_i v_i^{(*)}z_i(t)^2}{1+\gamma_i z_i(t)^2}.\end{aligned}$$

Next, we define the following sequence:

$$\bar{V}_i(t)=\mathbb{E}_{\varepsilon_i(t)}[v_i(t)],\ \text{and}\ \bar{V}_i(t+1)=\frac{\bar{V}_i(t)+\gamma_i v_i^{(*)}z_i(t)^2}{1+\gamma_i z_i(t)^2}.$$

By I.1 and I.2, the sequence $\{\bar{V}_i(t)\}_t$ is convergent, which is a contradiction with the divergence of $\{\mathbb{E}_{\varepsilon_i(t)}[v_i(t)]\}_t$. Therefore, the sequence $\left\{\bar{V}_i(t)\right\}_{t\geq 0}$ converges.

By using the fact that in both cases (i) and (ii), $\{\mathbb{E}_{\varepsilon_i(t)}[v_i(t)]\}_t$ is a convergent sequence, since

$$\bar{V}_i(t+1)=\frac{\bar{V}_i(t)+\gamma_i v_i^{(*)}z_i(t)^2}{1+\gamma_i z_i(t)^2}.$$

we have

$$\bar{V}_i(t+1)-v_i^{(*)}=\frac{\bar{V}_i(t)-v_i^{(*)}}{1+\gamma_i z_i(t)^2},\qquad\text{and}\qquad \bar{V}_i(t)-v_i^{(*)}=\frac{\bar{V}_i(0)-v_i^{(*)}}{\prod_{h=1}^{t-1}(1+\gamma_i z_i(t)^2)}.$$

Hence, we obtain that

$$\lim_{t\to+\infty}\mathbb{E}_{\varepsilon_i(t)}[v_i(t)]=\begin{cases} v_i^{(*)}, & \text{if } \lim_{t\to+\infty}z_i(t)\neq 0\\ v_i^{(*)}+\dfrac{v_i(0)-v_i^{(*)}}{\prod_{h=1}^{\infty}(1+\gamma_i z_i(t)^2)}, & \text{if } \lim_{t\to+\infty}z_i(t)=0.\end{cases}$$

□

Proposition 10

Proof. Conditioned on the input decisions $\mathcal{Y}(t)$, the MAP estimator follows a deterministic sequence:

$$\overline{v}_i(t+1)=\tilde{\mathtt{i}}_{i,t}\frac{\overline{v}_i(t)+\gamma_i v_i^{(*)}z_i(t)^2}{(1+\gamma_i z_i(t)^2)}. \tag{42}$$

By recursively substituting $\overline{v}_i(t)$ into $\overline{v}_i(t+1)$, we obtain:

$$\begin{aligned}
\overline{v}_i(1) &= \tilde{\mathtt{i}}_{i,0}\frac{v_i^{(0)}+\gamma_i v_i^{(*)}z_i(0)^2}{R_{i,0}},\\
\overline{v}_i(2) &= \tilde{\mathtt{i}}_{i,0}\tilde{\mathtt{i}}_{i,1}\frac{v_i^{(0)}+\gamma v_i^{(*)}z_i(0)^2}{R_{i,0}R_{i,1}}+\tilde{\mathtt{i}}_{i,1}\frac{\gamma_i v_i^{(*)}z_i(1)^2}{R_{i,1}},\\
\overline{v}_i(3) &= \tilde{\mathtt{i}}_{i,0}\tilde{\mathtt{i}}_{i,1}\tilde{\mathtt{i}}_{i,2}\frac{v_i^{(0)}+\gamma_i v_i^{(*)}z_i(0)^2}{R_{i,0}R_{i,1}R_{i,2}}+\tilde{\mathtt{i}}_{i,1}\tilde{\mathtt{i}}_{i,2}\frac{\gamma_i v_i^{(*)}z_i(1)^2}{R_{i,1}R_{i,2}}+\tilde{\mathtt{i}}_{i,2}\frac{\gamma_i v_i^{(*)}z_i(2)^2}{R_{i,2}},\\
\overline{v}_i(4) &= \tilde{\mathtt{i}}_{i,0}\tilde{\mathtt{i}}_{i,1}\tilde{\mathtt{i}}_{i,2}\tilde{\mathtt{i}}_{i,3}\frac{v_i^{(0)}+\gamma v_i^{(*)}z_i(0)^2}{R_{i,0}R_{i,1}R_{i,2}R_{i,3}}+\tilde{\mathtt{i}}_{i,1}\tilde{\mathtt{i}}_{i,2}\tilde{\mathtt{i}}_{i,3}\frac{\gamma_i v_i^{(*)}z_i(1)^2}{R_{i,1}R_{i,2}R_{i,3}}+\tilde{\mathtt{i}}_{i,2}\tilde{\mathtt{i}}_{i,3}\frac{\gamma_i v_i^{(*)}z_i(2)^2}{R_{i,2}R_{i,3}}+\tilde{\mathtt{i}}_{i,3}\frac{\gamma_i v_i^{(*)}z_i(3)^2}{R_{i,3}},\\
\vdots\quad &\quad\vdots\\
\overline{v}_i(t) &= v_i^{(0)}\left(\prod_{h=0}^{t-1}\frac{\tilde{\mathtt{i}}_{i,h}}{R_{i,h}}\right)+v_i^{(*)}\left(\gamma_i\sum_{h=0}^{t-1}z(h)^2\prod_{s=h}^{t-1}\frac{\tilde{\mathtt{i}}_{i,h}}{R_{i,s}}\right),
\end{aligned}$$

where $R_{i,t} = (1 + \gamma_i z_i(t)^2)$. This proves (26). We distinguish the following cases:

(1) If Ψ is a finite set, then there is a $t_0 \in \mathbb{N}$ such that for each $t \geq t_0$, we have $\overline{v}_i(t+1) = 0$ and therefore $\{\overline{v}_i(t)\}_t$ converges to 0.

(2) Assume that Ψ_i is an infinite set and $\lim_{t\to\infty} z_i(t) = 0$. Then, without loss of generality, let $\Psi_i \equiv \mathbb{N}$ (in other words, if $\Psi_i \not\equiv \mathbb{N}$, then there exists a bijection $\varphi : \mathbb{N} \to \varphi(\mathbb{N}) = \Psi_i$).

We now note that the convergence of $\{\overline{v}_i(t)\}_t$ is a direct consequence of the following result:

I.3 If $\overline{v}_i(0) \geq v_i^{(*)}$, then the sequence $\{\overline{v}_i(t)\}_t$ is decreasing and lower bounded by $v_i^{(*)}$.

I.4 If $\overline{v}_i(0) \leq v_i^{(*)}$, then the sequence $\{\overline{v}_i(t)\}_t$ is increasing and upper bounded by $v_i^{(*)}$.

In the case of I.3, we first prove that $\{\overline{v}_i(t)\}_t$ is lower bounded by $v_i^{(*)}$. We proceed by induction and let $\overline{v}_i(0) \geq v_i^{(*)}$. We show that if $\overline{v}_i(t) \geq v_i^{(*)}$, then $\overline{v}_i(t+1) \geq v_i^{(*)}$. By contradiction, let us assume that $\overline{v}_i(t+1) < v_i^{(*)}$. Then, we have

$$\frac{\overline{v}_i(t) + \gamma_i v_i^{(*)} z_i(t)^2}{(1 + \gamma_i z_i(t)^2)} < v_i^{(*)}.$$

This implies that $\overline{v}_i(t) < v_i^{(*)}$, which is a contradiction. Still in I.3, we can now prove that the sequence $\{\overline{v}_i(t)\}_t$ is decreasing:

$$\begin{aligned}\overline{v}_i(t+1) - \overline{v}_i(t) &= \frac{\overline{v}_i(t) + \gamma_i v_i^{(*)} z_i(t)^2}{(1 + \gamma_i z_i(t)^2)} - \overline{v}_i(t) \\ &= \gamma_i v_i^{(*)} z_i(t)^2 \frac{v_i^{(*)} - \overline{v}_i(t)}{(1 + \gamma_i z_i(t)^2)} \\ &\leq 0.\end{aligned}$$

In the case of I.4, we can use the same argument to prove that $\{\overline{v}_i(t)\}_t$ is increasing and upper bounded by $v_i^{(*)}$.

Using the proof of Proposition 9, we know that $\{\mathbb{E}_{\varepsilon_i(t)}[v_i(t)]\}_t$ is a convergent sequence, and that in the case $\lim_{t\to+\infty} z_i(t) = 0$ we have

$$\lim_{t\to+\infty} \overline{v}_i(t) = v_i^{(*)} + \frac{v_i(0) - v_i^{(*)}}{\prod_{h=1}^{\infty} (1 + \gamma_i z_i(h)^2)}.$$

(3) If Ψ is an infinite set and $\lim_{t\to\infty} z_i(t) \neq 0$, then we show that $\{\overline{v}_i(t)\}_t$ converges to $v_i^{(*)}$. In doing so, we first rewrite (42) as

$$\overline{v}_i(t+1) = \left(\frac{1}{1 + \gamma_i z_i(t)^2}\right) \left[\overline{v}_i(t) + \gamma_i v_i^{(*)} z_i(t)^2\right].$$

Next, we note that

$$\overline{v}_i(t+1) - v_i^{(*)} = \frac{\overline{v}_i(t) - v_i^{(*)}}{1 + \gamma_i z_i(t)^2}$$

implies the monotonicity of the sequence (namely, if $\overline{v}_i(0) \geq v_i^{(*)}$ then $\overline{v}_i(t+1) \leq \overline{v}_i(t)$ and if $\overline{v}_i(0) \leq v_i^{(*)}$ then $\overline{v}_i(t+1) \geq \overline{v}_i(t)$). Next, we have

$$\frac{\overline{v}_i(t+1) - v_i^{(*)}}{\overline{v}_i(t) - v_i^{(*)}} = \frac{1}{1 + \gamma_i z_i(t)^2} \leq 1.$$

Case 1) If the sequence $\{z_i(t)^2\}_t$ is divergent, then $\forall M > 0$, there exists $t_0 > 0$, such that $\forall t > t_0$, $z_i(t)^2 > M$. Therefore, for $M = 1$, $\exists t_0 > 0$, $\forall t > t_0$, $z_i(t)^2 > 1$. Then

$$\frac{1}{1 + \gamma_i z_i(t)^2} \leq \frac{1}{1 + \gamma_i}, \text{ for } t > t_0.$$

Let $\overline{\gamma}_i = \frac{1}{1+\gamma_i} < 1$, then we obtain

$$|\overline{v}_i(t + 1) - v_i^{(*)}| \leq \overline{\gamma}_i |\overline{v}_i(t) - v_i^{(*)}|.$$

This implies

$$|\overline{v}_i(t) - v_i^{(*)}| \leq \overline{\gamma}_i^t |v_i(0) - v_i^{(*)}| \underset{t\to\infty}{\longrightarrow} 0.$$

Case 2) If the sequence $\{z_i(t)^2\}_t$ converges to M, then by hypothesis

$$\lim_{t\to\infty} z_i(t) = M \neq 0.$$

As a consequence, $\forall \epsilon > 0$, $\exists t_0 > 0$ such that $\forall t > t_0$, $M - \epsilon \leq r^2(t) \leq M + \epsilon$. Choose an arbitrary $\epsilon > 0$ with $M - \epsilon > 0$. Then, $\exists t_0 > 0$ such that $\forall t > t_0$, $0 < M - \epsilon \leq r^2(t)$. Therefore

$$\frac{1}{1 + \gamma_i z_i(t)^2} \leq \frac{1}{1 + (M - \epsilon)\gamma_i}.$$

Let $\overline{\gamma}_i = \frac{1}{1+(M-\epsilon)\gamma_i} < 1$, then we obtain

$$|\overline{v}_i(t + 1) - v_i^{(*)}| \leq \overline{\gamma}_i |\overline{v}_i(t) - v_i^{(*)}|.$$

This implies that

$$|\overline{v}_i(t) - v_i^{(*)}| \leq \overline{\gamma}_i^t |v_i(0) - v_i^{(*)}| \underset{t\to\infty}{\longrightarrow} 0.$$

□

Lemma 3

Proof. The equilibrium prices for sectors in $\mathcal{N}_+$ and $\mathcal{N}_0$ are characterized separately. By definition of $\mathbb{E}_{\boldsymbol{\eta}}[r_i(t)]$, we replace the demand for labor and material input (40) into the Cobb-Douglas production and obtain

$$\hat{r}_i(t) = p_i(t + 1) q_i l_i(t)^{1-\phi} \prod_{s=1}^{n} y_{i,s}(t)^{\phi\delta_{i,s}}, \tag{43}$$

where $q_i = \exp(a_i + \sigma_i^2/2)$. From (40) and (43), we obtain

$$\hat{r}_i(t) = \overline{D}_i(t) \frac{p_i(t + 1)}{w(t + 1)^{1-\phi}} \hat{r}_i(t)^{1-\phi+\phi v_i}, \quad \text{where } \overline{D}_i(t) = q_i(1 - \phi)^{1-\phi} \left(\phi v_i \prod_{s=1}^{n} \left(\frac{\alpha_{i,s}}{p_s(t)}\right)^{\alpha_{i,s}}\right)^{\phi v_i}. \tag{44}$$

By (44), we obtain that if $v_i = 1$, then

$$p_i(t + 1) = \frac{w(t + 1)^{1-\phi}}{\overline{D}_i(t)}, \tag{45}$$

and if $v_i \neq 1$ and $v_i \neq 0$, then

$$\hat{r}_i(t) = \left(\frac{w(t+1)^{1-\phi}}{p_i(t+1)\overline{D}_i(t)}\right)^{\dfrac{1}{\phi(v_i-1)}}. \tag{46}$$

Plugging (40) into the market clearing conditions (5) and using the consumption demand (7) yields:

$$p_j(t)x_j(t-1) - \overline{w}(t)\kappa_j = \sum_{i=1}^{n} \phi v_i \alpha_{i,j} \hat{r}_i(t), \qquad \text{for } j \in \mathcal{N}. \tag{47}$$

Let us consider $\overline{x}_j(t) = p_j(t)x_j(t-1) - \overline{w}(t)\kappa_j$. In matrix form, (47) implies $\overline{\mathbf{x}}(t) = \phi\Delta^\top \hat{\mathbf{r}}(t)$. Hence,

$$\hat{\mathbf{r}}(t) = \frac{1}{\phi}(\Delta^\top)^{-1}\overline{\mathbf{x}}(t). \tag{48}$$

Since $\hat{r}_i(t) = p_i(t+1)q_i\mu_i(t)$, and by (45), (46) and (48), we obtain that the equilibrium prices and expected production, for each $i \in \mathcal{N}$:

$$p_i(t+1) = \frac{w(t+1)^{1-\phi}}{\overline{D}_i(t)}\left(\frac{\phi}{[(\Delta)^\top)^{-1}]_i\,\overline{\mathbf{x}}(t)}\right)^{\phi(v_i-1)},$$

$$\mu_i(t) = \frac{\overline{D}_i(t)}{w(t+1)^{1-\phi}}\left(\frac{\left[(\Delta^\top)^{-1}\right]_i\overline{\mathbf{x}}(t)}{\phi}\right)^{1+\phi(v_i-1)}.$$

Using (10) and the total labor supply condition (6) at the t-th period, the unitary salary reduces to:

$$w(t) = \begin{cases} 0, & \text{for } t = 1 \\ \left(\dfrac{1-\phi}{\phi}\right)\displaystyle\sum_{j=1}^{n}\hat{r}_j(t-1), & \text{for } t = 2,\dots,T. \end{cases} \tag{49}$$

If $v_i = 0$. Using (12) and (6), we have

$$\sum_{i\in\mathcal{N}_0}\left(\frac{(1-\phi)p_i(t+1)}{w(t+1)}\right)^{\frac{1}{\phi}} = 1 - \sum_{i\in\mathcal{N}_+} l_i(t),$$

with $\sum_{i\in\mathcal{N}_+} l_i(t)$ being determined in Lemma 2. We obtain the following characterization of equilibrium prices and expected production:[17]

$$p_i(t+1) \;=\; d_i \quad \text{where } d_i \text{ is the solution of } \sum_{i\in\mathcal{N}_0}\left(\frac{(1-\phi)d_i}{w(t+1)}\right)^{\frac{1}{\phi}} = 1 - \sum_{i\in\mathcal{N}_+} l_i(t), \tag{50}$$

$$\mu_i(t) \;=\; \left(\frac{(1-\phi)p_i(t+1)}{w(t+1)}\right)^{\frac{1-\phi}{\phi}}, \tag{51}$$

for each $i \in \mathcal{N}_0$. □

[17] Since $\lim_{z\to 0^+} z^z = 1$, we adopt the convention $0^0 = 1$.

Proposition 11

Proof. Using (1) and noticing that $\varepsilon_i(t) = \mathcal{N}(m_i, \sigma_i)$, we have

$$\begin{aligned}\mathbb{P}\left(v_i(t) = 0 \mid \mathcal{I}_i(t-1)\right) &= \mathbb{P}\left(\frac{v_i^{(0)} + \gamma_i \sum_{\ell=1}^t z_i(\ell) s_i(\ell)}{\left(1 + \gamma_i \sum_{\ell=1}^t z_i(\ell)^2\right)} \leq 0 \;\middle|\; \mathcal{I}_i(t-1)\right) \\ &= F\left(-\frac{\tau_i v_i^{(*)} \sum_{\ell=1}^t z_i(\ell)^2 + \sigma_i^2 \left(v_i^{(0)} + m_i \sum_{\ell=1}^t z_i(\ell)\right)}{\sigma_i^3 \sum_{\ell=1}^t z_i(\ell)^2}\right) \\ &= F\left(\tilde{t}(\sigma_i, \tau_i)\right),\end{aligned}$$

where F is the probability distribution function of a standardized Gaussian distribution and

$$\tilde{t}(\sigma_i, \tau_i) = -\frac{\tau_i v_i^{(*)}}{\sigma_i^3} - \frac{v_i^{(0)} + m_i \sum_{\ell=1}^t z_i(\ell)}{\sigma_i \sum_{\ell=1}^t z_i(\ell)^2}.$$

Note that there exists $\bar{\sigma}_i(\tau_i)$, such that for all $\sigma_i \leq \bar{\sigma}_i(\tau_i)$, we have:

$$\frac{\partial}{\partial \sigma_i} \tilde{t}(\sigma_i, \tau_i) = 3 \frac{\tau_i v_i^{(*)}}{\sigma_i^4} + \frac{v_i^{(0)} + m_i \sum_{\ell=1}^t z_i(\ell)}{\sigma_i^2 \sum_{\ell=1}^t z_i(\ell)^2} \geq 0, \quad \text{with} \quad \lim_{\sigma_i \to 0} \tilde{t}(\sigma_i, \tau_i) = -\infty.$$

Similarly,

$$\frac{\partial}{\partial \tau_i} \tilde{t}(\sigma_i, \tau_i) = -\frac{v_i^{(*)}}{\sigma_i^3} \leq 0, \quad \text{with} \quad \lim_{\tau_i \to +\infty} \tilde{t}(\sigma_i, \tau_i) = -\infty.$$

Since F is continuous, increasing and $\lim_{t \to -\infty} F(t) = 0$, then

$$\begin{cases} \lim_{\sigma_i \to 0} \mathbb{P}\left(v_i(t) = 0 \mid \mathcal{I}_i(t-1)\right) &= 0, \\ \lim_{\tau_i \to \infty} \mathbb{P}\left(v_i(t) = 0 \mid \mathcal{I}_i(t-1)\right) &= 0. \end{cases}$$

□

Proposition 12

Proof. Proposition 12 is obtained by replacing $v_i^{(0)}$ with $v_i(t)$ in Proposition 11, and $\tilde{z}_i^{(1)}(t)$ and $\tilde{z}_i^{(2)}(t)$ with $z_i(t)$ and $z_i(t)^2$, respectively. To lighten the exposition, we avoid reporting whole expressions extensively. □

Proposition 13

Proof. Based on the first-order conditions of the firm i's problem (see the proof of Lemma 2) and substituting $l_v(t)$ and $y_{i,j}(t)$ into the firm i's production, we note that $\hat{r}_i(t, v_i(t)) = \hat{r}_i(t, v_i^{(*)})$ is satisfied if and only if prices $\mathbf{p}(t+1)$ and $\mathbf{p}(t)$ for which

$$\hat{r}_i(t, v_i(t)) \;=\; p_i(t+1) \mathbb{E}_{\boldsymbol{\eta}}[\eta_i(t)] \left(\frac{(1-\phi)\hat{r}_i(t)}{w(t+1)}\right)^{(1-\phi)} \prod_{j \in \mathcal{N}} \left(\frac{\phi \delta_{i,j}(t) \hat{r}_i(t, v_i(t))}{p_j(t)}\right)^{\phi v_i^{(*)} \alpha_{i,j}}.$$

When $\sum_j \delta_{i,j}^{(*)} = \sum_j v_j^{(*)} \alpha_{i,j} = v^{(*)} = 1$, taking the logarithm yields

$$\log p_i(t+1) \;=\; \phi \sum_{j \in \mathcal{N}} \delta_{i,j}^{(*)} \log p_j(t) + k_i(t), \tag{52}$$

where

$$k_i(t) \;=\; -\,\phi \sum_{j \in \mathcal{N}} \delta_{i,j}^{(*)} \log \delta_{i,j}(t) + (1-\phi) \log w(t+1) - \log\left((1-\phi)^{1-\phi}\, \phi^{\phi}\right) - \log q_i,$$

We define the following matrices:

$$P(t) = \begin{bmatrix} \log p_i \\ \vdots \\ \log p_n(t) \end{bmatrix}, \qquad K(t) = \begin{bmatrix} k_1(t) \\ \vdots \\ k_n(t) \end{bmatrix}, \qquad \text{and} \qquad \overline{P}(t+1) = P(t+1) - P(t).$$

Then, (52) implies

$$\begin{aligned} \overline{P}(t+1) &= P(t+1) - P(t) \\ &= \left(\phi \Delta^{(*)} P(t) + K(t)\right) - \left(\phi \Delta^{(*)} P(t-1) + K(t-1)\right) \\ &= \left(\phi \Delta^{(*)}\right) \overline{P}(t) + \underbrace{K(t) - K(t-1)}_{U(t)}, \end{aligned}$$

so that

$$\overline{P}(t) = (\phi \Delta^{(*)})^{t-1} \overline{P}(1) + \sum_{h=0}^{t-2} (\phi \Delta^{(*)})^h U(t-1-h).$$

Since $\Delta^{(*)} \mathbf{1} = \mathbf{1}$ (which implies $(\Delta^{(*)})^h \mathbf{1} = \mathbf{1}$, where $\mathbf{1}$ is a vector of ones), and $\phi < 1$, we have

$$\begin{aligned} \lim_{t \longrightarrow \infty} \overline{P}(t) &= \lim_{t \longrightarrow \infty} \left[\sum_{h=0}^{t-2} (\phi \Delta^{(*)})^h U(t-1-h) \right] \\ &= \lim_{t \longrightarrow \infty} \left[(1-\phi) \sum_{h=0}^{t-2} \phi^h \mathbf{1} \log\left(\frac{w(t-h)}{w(t-h-1)}\right) \right], \end{aligned}$$

where we used the fact that

$$U_i(t) \;= \phi \textstyle\sum_{j \in \mathcal{N}} \delta_{i,j}^{(*)} \log \dfrac{\delta_{i,j}(t-1)}{\delta_{i,j}(t)} + (1-\phi) \log w(t+1) - (1-\phi) \log w(t),$$

so that, as long as $\Delta(t)$ converges, we have

$$U_i(t) \quad \rightsquigarrow \quad (1-\phi) \log w(t+1) - (1-\phi) \log w(t).$$

where $\rightsquigarrow$ denotes the limit behaviour. Formally, given functions $f(x)$ and $g(x)$, we have $f(x) \rightsquigarrow g(x)$ if and only if $\lim_{x \to \infty} \frac{f(x)}{g(x)} = 1$. Note that a sufficient condition for the existence of the limit is that there exists a constant $\xi > 0$ such that for each $t > t_0$, we have $w(t+1) \le \xi w(t)$, for some $t_0 \in \mathbb{N}$. Hence, we obtain

$$(\log \mathbf{p}(t+1) - \log \mathbf{p}(t)) = (1-\phi) \sum_{h=0}^{t-1} \phi^h \mathbf{1} \log\left(\frac{w(t-h)}{w(t-h-1)}\right),$$

which is equivalent to

$$\frac{p_i(t+1)}{p_i(t)} = \exp\left((1-\phi)W(t)\right), \quad \text{where} \quad W(t) = \sum_{h=0}^{t-2} \phi^h \log\left(\frac{w(t-h)}{w(t-h-1)}\right). \tag{53}$$

Hence, equation (52) provides the asymptotic behaviour of equilibrium prices for all $i \in \mathcal{N}$:

$$\log p_i(t+1) \;=\; \phi \sum_{j\in\mathcal{N}} v_i^{(*)} \alpha_{i,j} \log p_j(t) + \phi\left(h_i(t) - v_i^{(*)} \log v_i^{(*)}\right) + b(t), \qquad \text{for } i \in \mathcal{N},$$

where

$$h_i = -\frac{\log q_i}{\phi} - \sum_{j\in\mathcal{N}} \alpha_{i,j} \log \alpha_{i,j}, \quad b(t) = (1-\phi)\log w(t+1) - (1-\phi)\tilde{\phi}, \quad \text{and} \quad \tilde{\phi} = \frac{\log((1-\phi)^{1-\phi}\,\phi^{\phi})}{1-\phi}.$$

By considering the variation rates of individual prices in (53), we define

$$W(t) = \sum_{h=1}^{t-1} \phi^{t-h-1} \log\left(\frac{w(h+1)}{w(h)}\right), \quad \text{with} \quad W(t) = \log\left(\frac{w(t)}{w(t-1)}\right) + \phi W(t-1). \tag{54}$$

and replace $p_i(t+1) = \exp((1-\phi)W(t))p_i(t)$ into (52):

$$\begin{aligned} \log p_i(t) - \phi \sum_{j\in\mathcal{N}} v_i^{(*)} \alpha_{i,j} \log p_j(t) \quad &= \phi\left(h_i(t) + v_i^{(*)} \log v_i^{(*)}\right) + b(t) - (1-\phi)W(t) \\ (I - \phi A V^{(*)}) \log \mathbf{p}(t) \quad &= \phi\left(\tilde{\mathbf{h}} - V^{(*)} \log V^{(*)}\mathbf{1}\right) + (b(t) - (1-\phi)W(t))\mathbf{1} \end{aligned}$$

Solving with respect to $\log p_i(t)$, under the assumption that $v_i^{(*)} = 1$, we obtain

$$\begin{aligned} \log p_i(t) \quad &= \phi \sum_{j=1}^{n} \ell_{i,j}(h_j(t) - v_j^{(*)} \log v_j^{(*)}) + (b(t) - (1-\phi)W(t)) \sum_{j=1}^{n} \ell_{i,j} \\ &= \phi \sum_{j=1}^{n} \ell_{i,j} \left(\sum_{h=1}^{n} \alpha_{j,h} \log \alpha_{j,h}(t) + \frac{\log q_j}{\phi}\right) + \frac{b(t) - (1-\phi)W(t)}{1-\phi}, \end{aligned} \tag{55}$$

where ℓ_{vj} is the (v,j) element of the Leontief inverse, i.e., $L = (I - \phi A V^{(*)})^{-1}$. The second equality in (55) has been obtained by noticing that the property of constant returns to scale $v_i^{(*)} = 1$, for $i \in \mathcal{N}$, implies

$$L\mathbf{1} = (I - \phi A)^{-1}\mathbf{1} = \sum_{s=0}^{\infty} \phi^s (A)^s \mathbf{1} = \sum_{s=0}^{\infty} \phi^s \mathbf{1} = \frac{1}{1-\phi}\mathbf{1}.$$

We obtain

$$\begin{aligned} \log p_i(t) \quad &= \log w(t+1) + \phi \sum_{j=1}^{n} \ell_{i,j} h_j(t) - W(t) - \tilde{\phi} \\ &= \log w(t+1) + \phi \beta_i\left(\phi, A, \tilde{\mathbf{h}}\right) - W(t) - \tilde{\phi}, \end{aligned} \tag{56}$$

where $\beta_j(\phi, A, \boldsymbol{\zeta}) = \sum_{i\in\mathcal{N}} \ell_{i,j}\zeta_i$ is the generalized Bonacich centrality of the true input-output elasticity structure (i.e., the j-th element of $(I - \phi A^{\top})^{-1}\boldsymbol{\zeta}$). Combining (56) with (53) entails

$$\begin{aligned} p_i(t+1) \quad &= \exp\left((1-\phi)W(t)\right) p_i(t) \\ &= w(t+1) \exp\left(\phi\beta_i\left(\phi, A, \tilde{\mathbf{h}}\right) - \phi W(t) - \tilde{\phi}\right). \end{aligned}$$

□

Proposition 14

Proof. Under the assumptions of Proposition 13, using (27), we have

$$\begin{aligned}\left(\frac{\phi}{\left[\Delta^\top(t)^{-1}\right]_i \tilde{\mathbf{x}}(t)}\right)^{\phi(v_i-1)} &= w(t+1)^{\phi}\overline{D}_i(t)\exp\left(\phi\beta_i\left(\phi, A, \tilde{\mathbf{h}}\right) - \phi W(t) - \tilde{\phi}\right)\\ \frac{\phi}{\left[\Delta^\top(t)^{-1}\right]_i \tilde{\mathbf{x}}(t)} &= w(t+1)^{\frac{1}{v_i-1}}\left(\overline{D}_i(t)\right)^{\frac{1}{\phi(v_i-1)}}\exp\left(\frac{\beta_i\left(\phi, A, \tilde{\mathbf{h}}\right) - W(t) - \tilde{\tilde{\phi}}}{v_i-1}\right)\end{aligned} \tag{57}$$

where

$$\tilde{\tilde{\phi}} = \frac{\log((1-\phi)^{1-\phi}\,\phi^{\phi})}{\phi(1-\phi)}.$$

By replacing (56) into $\overline{D}_i(t)$, we have

$$\begin{aligned}\log \overline{D}_i(t) &= \log q_i(1-\phi)^{1-\phi} + \phi v_i\left(\sum_s \alpha_{i,s}\log\left(\frac{\phi v_i\alpha_{i,s}}{p_s(t)}\right)\right)\\ &= \phi h_i(t) - \phi v_i\sum_s \alpha_{i,s}\left(\log(w(t)) + \phi\beta_s\left(\phi, A, \tilde{\mathbf{h}}\right) - \phi W(t-1) - \tilde{\phi}\right)\\ &\quad +(1-\phi)\log(1-\phi) + \phi v_i\log\phi.\end{aligned}$$

Substituting this back into equation (57), we obtain

$$\begin{aligned}\frac{\phi}{\left[\Delta^\top(t)^{-1}\right]_i \tilde{\mathbf{x}}(t)} &= \left(\frac{w(t+1)}{w(t)^{v_i}}\right)^{\frac{1}{v_i-1}}\exp\left(\frac{1}{v_i-1}\left(\left[(I-\phi V(t)A)(I-\phi A)^{-1}\tilde{\mathbf{h}}\right]_i\right.\right.\\ &\quad \left.\left. -W(t) - \hat{\phi} + h_i(t) + \phi v_i W(t-1) + v_i(\log\phi + \tilde{\phi})\right)\right),\end{aligned}$$

where

$$\hat{\phi} = \frac{(1-\phi)\log(1-\phi)}{\phi} - \frac{\log((1-\phi)^{1-\phi}\,\phi^{\phi})}{\phi(1-\phi)} = -\log(1-\phi) - \frac{\log(\phi)}{1-\phi}.$$

Let

$$\iota_i(\{w(h)\}_{h=1}^t) = \exp\left(\frac{1}{1-v_i}\left(\left[M\tilde{\mathbf{h}}\right]_i - \hat{\phi} + v_i(\log\phi + \tilde{\phi})\right)\right)\left(\frac{w(t)^{v_i+1}}{w(t-1)}\right)^{\frac{1}{v_i-1}}\prod_{h=1}^{t-2}\left(\frac{w(h)}{w(h+1)}\right)^{\phi^{t-1-h}}.$$

Then the last equality becomes that

$$\begin{aligned}\frac{\left[\Delta^\top(t)^{-1}\right]_i \tilde{\mathbf{x}}(t)}{\phi} &= \frac{1}{w(t+1)^{\frac{1}{v_i-1}}}\iota_i\left(\{w(h)\}_{h=1}^t\right)\\ \frac{\phi}{1-\phi}w(t+1) &= \sum_{i=1}^n \frac{1}{w(t+1)^{\frac{1}{v_i-1}}}\iota_i\left(\{w(h)\}_{h=1}^t\right).\end{aligned}$$

□

Proposition 15

Proof. Let us rewrite (32) as $w = g_t(w)$, where

$$g_t(w) = \sum_{i=1}^n \frac{1}{w^{\frac{1}{v_i-1}}}\tilde{\iota}_i, \quad \text{with} \quad \tilde{\iota}_i = \frac{1-\phi}{\phi}\iota_i\left(\{w(h)\}_{h=1}^{t-1}\right) \geq 0.$$

Note that $g_t'(w)$ is continuous and strictly increasing, independently from the values of $v_1, \ldots, v_n$:

$$g_t'(w) = \sum_{i=1}^{n} \left(-\frac{\tilde{\iota}_i}{v_i - 1} \right) \frac{1}{w^{\frac{v_i}{v_i - 1}}},$$

We study two cases.

First case. If $v_i > 1$ for all $i \in \mathcal{N}$, then $g_t'(w)$ is negative and $g_t(w)$ is continuous and strictly decreasing. We have the following facts:

$$\begin{cases} \lim_{w \to 0} g_t(w) & = \infty \\ \\ \lim_{w \to \infty} g_(w) & = 0 \end{cases} \quad \text{and} \quad \begin{cases} \lim_{w \to 0} g_t'(w) & = -\infty \\ \\ \lim_{w \to \infty} g_t'(w) & = 0. \end{cases}$$

Therefore, by the intermediate value theorem g_t admits a unique fixed-point.

Second case. If $v_i < 1$ for all $i \in \mathcal{N}$, then $g_t'(w)$ is positive and $g_t(w)$ is continuous and strictly increasing. We have the following facts:

$$\begin{cases} \lim_{w \to 0} g_t(w) & = 0 \\ \\ \lim_{w \to \infty} g_(w) & = \infty \end{cases} \quad \text{and} \quad \begin{cases} \lim_{w \to 0} g_t'(w) & = 0 \\ \\ \lim_{w \to \infty} g_t'(w) & = \infty. \end{cases}$$

Based on the continuity and monotonicity of $g_t'(w)$, as well as the above limit properties, we have that $w = 0$ is the unique fixed point of g_t.

□

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