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Robust Nonparametric Partial Frontier Input Distance Function Estimates: Testing and Rejecting Convexity

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Abstract

Statistical tests of the convexity assumption for full frontier technologies are still rather rarely applied. We employ existing order-α and order-m partial frontier estimators for production technologies. Since these partial frontier technology estimators have favourable statistical properties, this permits testing convexity using a simple statistical test comparing densities. Using a panel of electric power generation plants in the US we find strong evidence against convexity for most years using both order-α and order-m partial production frontiers. These new empirical results imply that empirical researchers must be cautious about invoking the –mostly implicit– convexity assumption when estimating technologies. Anyway, it is clear that convexity needs to be tested.

Keywords: Convexity, Technology, Robust frontier, Partial frontier, Order-m, Order-α.
JEL classification: C12, D24.

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1 Introduction

The technology is a basic empirical tool to analyze a range of interesting questions for managers, policy makers, and regulators. Examples include the determination of returns to scale, measuring efficiency in a static context, determining productivity change over time, etc. Historically, technology has almost always been assumed to be convex. However, already Farrell (1959) recognized that various factors may cause nonconvexities in technology. Nonconvexities in technology are primarily caused by indivisibilities in inputs and in outputs, the presence of increasing returns to scale, negative and positive externalities, etc. (see an overview in Briec, Kerstens, and Van de Woestyne (2022)). Shephard (1970, p. 15) stresses that convexity essentially implies a perfect time divisibility assumption: thus, if a production process is not perfectly time divisible and it is characterised by some minimal setup times (and hence setup costs), then the convexity assumption does not apply. Thus, convexity is an empirical assumption that ideally requires testing.

The wide majority of studies using parametric, semi-parametric or nonparametric specifications of production possibilities sets also maintains –implicitly or explicitly– that technology is convex. Statistical tests for the convexity of the technology have been available for some time (see Kneip, Simar, and Wilson (2016), Simar and Wilson (2020) and Simar and Wilson (2025)) and have been rather widely applied in empirical studies. Evidence rejecting convexity has been documented in a variety of contexts, including USA (Wilson (2021)) and Chinese (Wilson and Zhao (2023)) banking sectors, high-performance computing capabilities (Apon, Ngo, Payne, and Wilson (2015)), USA municipalities (O'Loughlin and Wilson (2021)), the UK independent school sector (Lopez-Torres, Johnes, Elliott, and Polo (2021)), a macro-economic cross-country analysis covering 143 countries (Wilson and Zhao (2025)), and Swedish district courts (Chen, Kerstens, and Zhu (2025)), among others. One study lists Belgian train traffic control centers (Kerstens, Roets, Van de Woestyne, and Zhao (2026)), and several other secondary data sets (two electricity and one agricultural data sets) for which convexity is also rejected. Sometimes, convexity fails rejection: Kerstens, Roets, Van de Woestyne, and Zhao (2026) lists a car registration and a dairy farm data set; Martínez Córdoba, Ramos, Zafra Gómez, and Zafra Gómez (2025) evaluate environmental protection spending for 120 countries; Schubert, Kroll, and Garcia Chavez (2023) analyse 104 German full universities; etc. Kapelko, Ortiz, and Aparicio (2025) analyse a composite corporate social responsibility indicator for food and beverage companies across regions of Europe, USA, Canada and Asia–Pacific countries over the period 2009–2018: convexity is rejected for the majority of years, but not for all years.

Another partially mixed picture is found in the semiconductor industry where for only three out of 20 years convexity cannot be rejected (Qiao and Wang (2021)). Since the testing of convexity of technology is a timely and still understudied topic, in this paper we present a test of convexity for another class of frontier methodologies.

Researchers often expect that eventual nonconvexities do not have any impact on economic value functions (like the cost function). Kerstens and Zhao (2025) are the first to specify a test for the convexity of nonparametric specifications of the cost function in the outputs. These authors provide evidence that cost functions are often nonconvex in the outputs which translates into nonconvex technologies. Kerstens, O'Donnell, Van de Woestyne, and Zhao (2026) reject convexity of the cost function in the outputs for Chinese but not for USA banks.

The purpose of this contribution is to re-assess these tests for the convexity of the production technology using robust partial frontier models. Partial frontiers allow for some observations to be situated beyond the boundary of the estimated technology, in contrast to traditional frontier estimators. It is common to distinguish between order-m and order-α partial frontier technologies. Cazals, Florens, and Simar (2002) is the seminal article proposing a nonconvex order-m technology. The article by Daraio and Simar (2007b) defines for the first time convex order-m technologies. Aragon, Daouia, and Thomas-Agnan (2005) are the first to define order-α estimators for nonconvex technologies, while Daouia and Simar (2007), Daraio and Simar (2007a) and Simar and Vanhems (2012) offer further developments. Convex order-α technologies have been first proposed in Ferreira and Marques (2020), whereby the articles by Polemis, Stengos, Tzeremes, and Tzeremes (2021) and Dai, Kuosmanen, and Zhou (2023) offer some complementary perspectives. While convex and nonconvex order-m and order-α technologies have been around for some time, to our knowledge nobody has cared yet to define a test of convexity for these partial frontier technologies.

The survey by Simar and Wilson (2015) partially discusses the statistical properties of both order-m and order-α partial frontiers. A key statistical property of both order-m and order-α partial frontier technologies is that both families of methods are known to be $\sqrt{n}$ consistent (see Wheelock and Wilson (2013) and Simar and Wilson (2015)). In this context, it is possible to employ a simple distribution test comparing density estimates like the Li-test (see Li, Maasoumi, and Racine (2009) for the most recent version). Compared to traditional frontiers methods, these two methods are resistant to outliers by using partial frontiers which are positioned slightly below the deterministic full frontiers (e.g., see Daouia and Simar (2007, Figure 1) for an example). Thus, a partial frontier approach mitigates the influence of outliers, unlike methods that rely on full frontiers when approximating the true frontiers.

Examples of empirical applications of partial frontiers are, e.g., Walden and Tomberlin (2010) on USA fishing capacity estimation and Varabyova, Blankart, Torbica, and Schreyögg (2017) on a two countries comparison of hospital performance, among others. Some studies compare both families of partial frontiers estimations: see, e.g., Krüger (2014). In terms of empirical applications, it is rather straightforward to establish that the slightly older order-m partial frontiers are quite a bit more popular than the more recent order-α partial frontiers.[1] A cursory reading of the literature indicates that the large majority of empirical applications employs nonconvex partial technologies.[2]

Based on a panel of U.S. electric power generation plants compiled by Kumbhakar and Tsionas (2011), our empirical analysis finds compelling evidence rejecting convexity for most years when evaluating both order-m and order-α partial production frontiers. These results highlight that researchers should exercise caution before imposing the commonly assumed –often implicit– convexity restriction in empirical technology estimation. We recommend that the assumption of convexity is explicitly tested rather than taken for granted.

2 Production Technologies and Their Representations

2.1 Production Technology and Distance Function

We begin by introducing the relevant definitions of technology. Let $x \in \mathbb{R}_+^N$ denote a vector of N input quantities and let $y \in \mathbb{R}_+^M$ denote a vector of M output quantities. The production possibility set or technology is then defined as follows:

$$\mathcal{T} := \{(x, y) \in (\mathbb{R}_+^N, \mathbb{R}_+^M) \mid x \text{ can produce at least } y\}. \tag{1}$$

[1]A search on Google Scholar on March 12, 2026 with the keywords "`partial frontier order-m`" yields about 7450 hits, while a search with the keywords "`partial frontier order-alpha`" yields just about 588 hits.

[2]We only have indirect evidence to substantiate this claim: the only ready made packages in `R` and `Stata` available to our knowledge provide only nonconvex models. Indeed, the `R`-packages known to us are `frontiles` and `FEAR` : they have only functions for nonconvex models. Equally so, the only `Stata`-package for partial frontiers described in Tauchmann (2012) only contains nonconvex models. The convex partial models developed in this paper are implemented based on `FEAR`. All codes and data used in the empirical illustration are publicly available on GitHub at `https://github.com/srzhao89/convex-partial-technology`.

For a given observation (x_0, y_0), the popular (inverse of the) input distance function (e.g., Färe (1988, p. 35)) is defined as follows:

$$\mathcal{E}(x_0, y_0) := \inf \{\mathcal{E} > 0 \mid (\mathcal{E}x_0, y_0) \in \mathcal{T}\}. \tag{2}$$

Following the seminal contribution of Cazals, Florens, and Simar (2002), the technology $\mathcal{T}$ can alternatively be characterized within a probabilistic framework. Let (X, Y) be a random vector whereby the input vector $X \in \mathbb{R}_+^N$ and the output vector $Y \in \mathbb{R}_+^M$. One can now define the joint distribution function $H_{XY}(x, y)$ as follows:

$$H_{XY}(x, y) = \mathrm{P}(X \leq x, Y \geq y) = \mathrm{P}(X \leq x \mid Y \geq y)\mathrm{P}(Y \geq y). \tag{3}$$

Denote

$$F_{X|Y}(x \mid y) = \mathrm{P}(X \leq x \mid Y \geq y), \tag{4}$$

and

$$S_Y(y) = \mathrm{P}(Y \geq y), \tag{5}$$

which are the conditional distribution function on X and the marginal survivor function on Y, respectively. The technology $\mathcal{T}$ can then be equivalently expressed in terms of this joint distribution function as follows:

$$\mathcal{T} = \{(x, y) \mid H_{XY}(x, y) > 0\}. \tag{6}$$

Therefore, the input distance function $\mathcal{E}(x_0, y_0)$ can now be equivalently expressed as

$$\mathcal{E}(x_0, y_0) = \inf \left\{\mathcal{E} > 0 \mid F_{X|Y}(\mathcal{E}x_0, y_0) > 0\right\}, \tag{7}$$

provided that $S_Y(y_0) > 0$.

2.2 Robust Input Distance Functions

We next introduce the concept of partial frontiers, which relax the assumption that all observations must lie within the estimated production possibility set and therefore permit a subset of observations to fall outside the frontier. Within this framework, we differentiate between two classes of partial frontier technologies, namely the order-α and order-m partial frontiers.

We begin with the order-α partial frontier, which is constructed such that a proportion $\alpha \times 100$

percent of observations are allowed to lie beyond the boundary of the estimated partial frontier. In other words, the frontier is defined relative to a conditional quantile of the technology rather than its extreme boundary. Formally, following Daouia and Simar (2007) the corresponding nonconvex order-α input distance function is defined as follows:

$$\mathcal{E}_{nc,\alpha}(x_0, y_0) = \inf \left\{ \mathcal{E} > 0 \mid F_{X|Y}(\mathcal{E}x_0, y_0) > 1 - \alpha \right\}. \tag{8}$$

An estimator of the convexified order-α input distance function is obtained as follows:

$$\begin{aligned} \widehat{\mathcal{E}}_{c,\alpha}(x_0, y_0) = \min_{\mathcal{E}, w_1, \ldots, w_n} \Big\{ & \mathcal{E} \mid y_0 \leq \sum_{i=1}^{n} w_i Y_i,\ \mathcal{E}x_0 \geq \sum_{i=1}^{n} w_i X_i \widehat{\mathcal{E}}_{nc,\alpha}(X_i, Y_i),\ \mathcal{E} \geq 0, \\ & \sum_{i=1}^{n} w_i = 1,\ \forall\ w_i \geq 0 \Big\}, \end{aligned} \tag{9}$$

where $\widehat{\mathcal{E}}_{nc,\alpha}(X_i, Y_i)$ is the estimate of the nonconvex order-α input distance function for (X_i, Y_i), i.e., it is the estimate of

$$\mathcal{E}_{nc,\alpha}(X_i, Y_i) = \inf \left\{ \mathcal{E} > 0 \mid F_{X|Y}(\mathcal{E}X_i, Y_i) > 1 - \alpha \right\}, \tag{10}$$

and $X_i \widehat{\mathcal{E}}_{nc,\alpha}(X_i, Y_i)$ is the estimated efficient input on the nonconvex order-α partial production frontier to produce the output Y_i.

Second, the order-m input distance function is based on an alternative partial frontier. For a fixed output level y_0, let $X_1, \ldots, X_m$ be independent and identically distributed random draws from the conditional distribution $F_{X|Y}(x \mid y_0)$. These m sampled inputs are then used to construct the order-m random technology, which characterizes the technology of units producing at least the output level y_0:

$$\mathcal{T}_m(y_0) = \left\{ (x, y) \in \mathbb{R}_+^{N+M} \mid x \geq X_i, y \geq y_0 \right\}. \tag{11}$$

The input distance function relative to the set $\mathcal{T}_m(y_0)$ is:

$$\tilde{\mathcal{E}}_m(x_0, y_0) = \inf \left\{ \mathcal{E} > 0 \mid (\mathcal{E}x_0, y_0) \in \mathcal{T}_m(y_0) \right\}. \tag{12}$$

The nonconvex order-m input distance function is then defined as:

$$\mathcal{E}_{nc,m}(x_0, y_0) = E_{X|Y}(\tilde{\mathcal{E}}_m(x_0, y_0) \mid Y \geq y_0). \tag{13}$$

Here, $E_{X|Y}$ denotes the expectation with respect to the conditional distribution $F_{X|Y}(x \mid y_0)$. The order-m input distance function assesses a unit at (x_0, y_0) by comparing its input vector to the expected minimum input among m randomly drawn peers, each of which produces at least the output level y_0.

An estimator of the convexified order-m input distance function is now obtained as follows:

$$\widehat{\mathcal{E}}_{c,m}(x_0, y_0 \mid w_0) = \min_{\mathcal{E}, w_1, \ldots, w_n} \Big\{ \mathcal{E} \mid y_0 \leq \sum_{i=1}^{n} w_i Y_i,\ \mathcal{E} x_0 \geq \sum_{i=1}^{n} w_i X_i \widehat{\mathcal{E}}_{nc,m}(X_i, Y_i),\ \mathcal{E} \geq 0, \sum_{i=1}^{n} w_i = 1,\ \forall\, w_i \geq 0 \Big\}, \tag{14}$$

where $\widehat{\mathcal{E}}_{nc,m}(X_i, Y_i)$ is the estimate of the nonconvex order-m input distance function for (X_i, Y_i), i.e., it is the estimate of

$$\mathcal{E}_{nc,m}(X_i, Y_i) = E_{X|Y}(\tilde{\mathcal{E}}_m(X_i, Y_i) \mid Y \geq Y_i), \tag{15}$$

Thus, $X_i \widehat{\mathcal{E}}_{nc,m}(X_i, Y_i)$ is the estimated efficient input on the nonconvex order-m production frontier to produce the given output Y_i.

2.3 Input Distance Functions on Full Frontiers

To facilitate conceptual comparison with the aforementioned partial frontier technologies, we now finally introduce the concept of full frontier technologies under the variable returns to scale (VRS) assumption and imposing either convexity or nonconvexity, respectively.[3]

The estimator of a convex input distance function on a full VRS frontier is given as follows:

$$\widehat{\mathcal{E}}_{c}(x_0, y_0) = \min_{\theta, w_1, \ldots, w_n} \Big\{ \theta \mid \theta x_0 \geq \textstyle\sum_{i=1}^{n} w_i X_i,\ y_0 \leq \sum_{i=1}^{n} w_i Y_i,\ \theta \geq 0, \displaystyle\sum_{i=1}^{n} w_i = 1,\ \forall\, w_i \geq 0 \Big\}. \tag{16}$$

This estimator (16) is the full frontier counterpart of the partial convex order-α estimator (9) and the partial convex order-m estimator (14). The corresponding estimator of nonconvex input distance function on a full VRS frontier is defined as:

$$\widehat{\mathcal{E}}_{nc}(x_0, y_0) = \min_{\theta, w_1, \ldots, w_n} \Big\{ \theta \mid \theta x_0 \geq \textstyle\sum_{i=1}^{n} w_i X_i,\ y_0 \leq \sum_{i=1}^{n} w_i Y_i,\ \theta \geq 0, \displaystyle\sum_{i=1}^{n} w_i = 1,\ \forall\, w_i \in \{0, 1\} \Big\}. \tag{17}$$

[3] In the operations research literature, these convex and nonconvex full frontier technologies are commonly known as Data Envelopment Analysis (DEA) and Free Disposal Hull (FDH) estimators, respectively.

This estimator (17) is the full frontier counterpart of the partial convex order-α estimator (10) and the partial convex order-m estimator (15).

3 The Li-test for Comparing Densities

To formally assess differences between two distributions of input distance function values, we employ a nonparametric test originally proposed by Li (1996) and subsequently refined by Fan and Ullah (1999) and more recently by Li, Maasoumi, and Racine (2009). By contrast to tests that focus on specific moments, this approach evaluates differences across the entire distribution. Specifically, it tests the statistical significance of differences between two kernel-based density estimates f and g of a random variable x. While the null hypothesis states that the two densities are equal almost everywhere ($H_0 : f(x) = g(x)\ \forall x$), the alternative hypothesis allows for differences at some points ($H_1 : f(x) \neq g(x)$ for some x). Crucially, the test is valid for both dependent and independent samples, which is important because frontier estimators –such as the input distance functions– typically exhibit dependency induced by the sample. In our application, we report exact p-values computed using 1,000 bootstrap replications. As usual, a large p-value indicates that the null hypothesis cannot be rejected, whereas a small p-value provides evidence against the null hypothesis.

In this study, we compare input distance function values for partial frontiers which have no fixed upper bound (unlike in traditional full frontiers). Consequently, there is no need to adjust this Li-test statistic to account for spurious boundary mass, as discussed in Simar and Zelenyuk (2006).

4 Empirical Illustration: US Electric Power Generation Plants

This Li-test is applied to a secondary data set of electric power generation plants in the US compiled by Rungsuriyawiboon and Stefanou (2007) and made publicly available by Kumbhakar and Tsionas (2011). This sample contains a total of $1,065$ yearly plant observations over the years 1986 to 1998.[4] The electric power generation plants available in our sample utilize three inputs: (i) capital (X_1), (ii) labor (X_2), and (iii) fuel (X_3). These inputs jointly produce the single output net

[4] The Kumbhakar and Tsionas (2011) article mentions the existence of a larger data set: the initial article of Rungsuriyawiboon and Stefanou (2007) also does contain the additional year 1999. However, the data available on the Journal of Applied Econometrics data archive is more limited: see `http://qed.econ.queensu.ca/jae/2011-v26.2/kumbhakar-tsionas/`.

steam electric power generation, measured in megawatt-hours (Y). This output represents the electricity generated from fossil-fuel-fired boilers that produce steam for turbine generators over a given observation period. Detailed definitions of the inputs and the output, as well as the construction of the series and descriptive statistics, is found in Rungsuriyawiboon and Stefanou (2007, p. 230–231). For each year in the sample, there are 82 observations, with the exception of our last year 1998 for which only 81 observations are available.

Table 1: Mean Input Distance Function Estimates under Robust Convex and Nonconvex Technologies: Based on Order-α Frontier where $\alpha = 0.95$

			—— Li-Test ——	
Year	$\widehat{\mu}_{c,\alpha}$	$\widehat{\mu}_{nc,\alpha}$	Statistics	p-values
1986	1.123	1.290	2.227	0.002
1987	1.147	1.327	3.848	0.000
1988	1.120	1.308	-0.180	0.007
1989	1.144	1.361	1.722	0.001
1990	1.078	1.253	2.070	0.003
1991	1.105	1.298	1.585	0.003
1992	1.085	1.287	-0.213	0.008
1993	1.136	1.303	1.181	0.012
1994	1.097	1.281	-2.969	0.075
1995	1.061	1.247	3.689	0.052
1996	1.037	1.260	0.999	0.000
1997	1.042	1.262	1.155	0.001
1998	1.130	1.352	12.362	0.145

When constructing the order-α and order-m partial frontiers, we set $\alpha = 0.95$ and we define m as the largest integer not exceeding αn (i.e., $0.95 \cdot 82 = 77.9$, so the floor is $m = 77$), where n is the number of observations. The partial frontier empirical results are reported in Table 1 for the order-α frontier and in Table 2 for the order-m frontier. In each table, the first column reports the year, the second and third columns present the average convex and nonconvex input distance function estimates, respectively, and the fourth and fifth columns display the Li-test statistic along with its associated p-value. In Table 1 we observe that as expected the average order-α convex input distance function $\widehat{\mu}_{c,\alpha}$ is always lower than or equal to the nonconvex counterpart $\widehat{\mu}_{nc,\alpha}$. Furthermore, both estimates are always slightly above unity. Finally, the Li-test rejects the null hypothesis of a common distribution for almost all years, except for the last year 1998. With regard to the order-m results, Table 2 shows that the average order-m convex input distance function $\widehat{\mu}_{c,m}$

Table 2: Mean Input Distance Function Estimates under Robust Convex and Nonconvex Technologies: Based on Order-m Frontier where $m = 77$

			—— Li-Test ——	
Year	$\widehat{\mu}_{c,m}$	$\widehat{\mu}_{nc,m}$	Statistics	p-values
1986	0.895	1.005	5.844	0.000
1987	0.920	1.028	6.605	0.000
1988	0.886	1.017	8.493	0.000
1989	0.885	1.031	7.988	0.000
1990	0.859	0.992	9.448	0.000
1991	0.889	1.001	7.082	0.000
1992	0.890	1.002	4.330	0.000
1993	0.889	1.003	2.479	0.000
1994	0.878	0.998	3.663	0.000
1995	0.870	0.984	5.644	0.000
1996	0.881	0.986	3.302	0.000
1997	0.875	0.988	3.366	0.000
1998	0.909	1.014	3.760	0.000

Table 3: Mean Input Distance Function Estimates under Full Convex and Nonconvex Technologies

			——KSW-Test#1 ——		——KSW Test#2 ——	
Year	μ_c	μ_{nc}	Statistics	p-values	Statistics	p-values
1986	0.820	0.938	0.359	0.751	0.175	0.756
1987	0.830	0.956	0.827	0.226	0.328	0.209
1988	0.805	0.953	0.712	0.203	0.293	0.199
1989	0.808	0.960	1.090	0.102	0.333	0.264
1990	0.790	0.934	0.669	0.104	0.243	0.197
1991	0.809	0.938	0.689	0.154	0.237	0.317
1992	0.806	0.936	0.552	0.190	0.260	0.153
1993	0.795	0.931	0.652	0.082	0.294	0.044
1994	0.790	0.927	0.588	0.096	0.275	0.067
1995	0.790	0.928	0.632	0.035	0.261	0.047
1996	0.802	0.924	0.417	0.254	0.225	0.198
1997	0.796	0.922	0.476	0.187	0.197	0.288
1998	0.819	0.935	0.701	0.053	0.276	0.070

estimates are again always lower than their nonconvex counterparts $\widehat{\mu}_{nc,m}$. Moreover, the convex order-m estimates now remain strictly below unity across all years. Finally, the Li-test statistics decisively reject the null hypothesis of a common distribution for all years. In Appendix A we provide a sensitivity analysis whereby we set $\alpha = 0.975$ and the corresponding $m = 79$: the overall conclusions regarding the rejection of convexity do not change at all.

Table 3 reports the results of the convexity tests for the full frontier technologies developed by Kneip, Simar, and Wilson (2016) and extended by Simar and Wilson (2020). The test proposed by Kneip, Simar, and Wilson (2016) is based on a comparison of the sample means of the nonconvex and convex input distance function estimators denoted by μ_{nc} and μ_c, respectively. Under the null hypothesis of convexity, these two means are equal ($H_0 : \mu_{nc} = \mu_c$), whereas the alternative hypothesis maintains that the nonconvex estimator has a larger mean than its convex counterpart ($H_1 : \mu_{nc} > \mu_c$).

Because this test relies on the comparison of independent sample means, the original sample is randomly partitioned into two independent subsamples. The resulting one-sided test statistic is asymptotically normally distributed. However, the outcome of the test may depend on the particular random split of the sample. To address this issue, Simar and Wilson (2020) propose repeating the procedure over multiple random splits and aggregating the information across splits using bootstrap techniques. Specifically, they introduce two complementary testing procedures: one procedure is based on the average of the test statistics computed across multiple splits (referred to as KSW-Test#1), and another procedure is based on a Kolmogorov–Smirnov test assessing the uniformity of the p-values obtained from each split (referred to as KSW-Test#2).

In our implementation, we follow the recommendations of Simar and Wilson (2020) and employ 100 random sample splits, 1,000 bootstrap replications to approximate the sampling distributions of the two test statistics under the null hypothesis of convexity, and an additional 100 bootstrap replications to estimate the bias correction term. Further methodological details are found in Simar and Wilson (2020).

Table 3 indicates that the null hypothesis of convexity is rejected in 4 years out of the 13 years considered by the two KSW tests at the 10% significance level. These results suggest that convexity is unlikely to hold uniformly across the years covered in this sample. Although the evidence against convexity is weaker than the one obtained from the Li-tests for the above robust partial frontiers, the rejection of convexity for the nonparametric production technology is consistent with the findings of Kerstens and Zhao (2025) who use the same data set and also report strong evidence against convexity for nonparametric cost functions in outputs for most years in their study.

5 Conclusions

Convexity of the production technology is a widely adopted –but often implicit– assumption in empirical studies across economics and related fields. However, whether this assumption holds in practice is ultimately an empirical question that requires formal testing with appropriate statistical tools. To our knowledge, Kneip, Simar, and Wilson (2016) are the first to propose a statistical test for the convexity of nonparametric production technology specifications.

In this study, we extend their approach to the context of partial order-α and order-m production frontiers by proposing a simple Li-test: this is possible because of the $\sqrt{n}$ consistency of the partial frontier input distance function estimates. Using a secondary data set of US electric power generation plants from Rungsuriyawiboon and Stefanou (2007), we find that convexity is uniformly rejected for most years in the sample when applying this simple Li-test. These findings hold consistently for both order-α and order-m partial production frontiers, and remain robust under modest variations of the parameters α and m.

By doing so, we provide practitioners with new tools to empirically assess the convexity of production technologies in the context of partial frontiers. We encourage researchers to critically evaluate the widespread assumption of convexity by applying either the framework developed here for robust partial frontiers, or the Kneip, Simar, and Wilson (2016) tests for traditional full frontiers.

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Appendix A Sensitivity Analysis

As a sensitivity analysis, we also consider $\alpha = 0.975$ and m is the largest integer that is no more than αn (i.e., $0.975 \cdot 82 = 79.95$ whose floor is 79), where n is the number of observations. The empirical results are presented in Table 4 for order-α and Table 5 for order-m. The structure of these two Tables 4 and 5 is similar to the Tables 1 and 2, respectively.

Table 4 reveals that the average order-α convex input distance function $\widehat{\mu}_{c,\alpha}$ is always lower than the nonconvex counterpart $\widehat{\mu}_{nc,\alpha}$. The Li-test convincingly rejects the null hypothesis of a common distribution, except for one year (i.e., 1998). Table 5 shows that the average order-m convex input distance function $\widehat{\mu}_{c,m}$ is clearly always lower than its nonconvex counterpart $\widehat{\mu}_{nc,m}$. Once more, the Li-test rejects overwhelmingly the null hypothesis of a common distribution.

Table 4: Mean Input Distance Function Estimates under Robust Convex and Nonconvex Technologies: Based on Order-α Frontier where $\alpha = 0.975$

			—— Li-Test ——	
Year	$\widehat{\mu}_{c,\alpha}$	$\widehat{\mu}_{nc,\alpha}$	Statistics	p-values
1986	0.957	1.111	2.995	0.000
1987	1.013	1.136	1.776	0.001
1988	0.925	1.127	3.273	0.000
1989	0.947	1.146	3.955	0.000
1990	0.902	1.081	2.975	0.000
1991	0.929	1.097	2.581	0.000
1992	0.932	1.107	3.245	0.000
1993	0.943	1.126	1.284	0.000
1994	0.949	1.119	1.011	0.000
1995	0.893	1.078	6.748	0.001
1996	0.929	1.083	1.507	0.000
1997	0.942	1.095	0.569	0.002
1998	1.006	1.182	8.063	0.157

Table 5: Mean Input Distance Function Estimates under Robust Convex and Nonconvex Technologies: Based on Order-m Frontier where $m = 79$

			—— Li-Test ——	
Year	$\widehat{\mu}_{c,m}$	$\widehat{\mu}_{nc,m}$	Statistics	p-values
1986	0.893	1.002	6.122	0.000
1987	0.917	1.025	6.883	0.000
1988	0.883	1.014	8.791	0.000
1989	0.883	1.028	8.309	0.000
1990	0.857	0.990	9.727	0.000
1991	0.887	0.999	7.408	0.000
1992	0.887	1.000	4.507	0.000
1993	0.886	1.001	2.596	0.000
1994	0.875	0.995	3.792	0.000
1995	0.867	0.982	5.789	0.000
1996	0.879	0.984	3.380	0.000
1997	0.872	0.986	3.451	0.000
1998	0.906	1.011	3.979	0.000