June 2026

WORKING PAPER SERIES

2026-EQM-04

Testing Cost Function Convexity under Nonconvex Variable Returns-to-Scale Technologies: Evidence from U.S. Banks

Xiaoqing CHEN
School of Management Science and Engineering, Nanjing University of Information Science and Technology, Nanjing, 210044, China. 003694@nuist.edu.cn

Kristiaan KERSTENS
Univ. Lille, CNRS, IESEG School of Management, UMR 9221 - LEM - Lille Économie Management, F-59000 Lille, France. k.kerstens@ieseg.fr. Tel: +33 320545892 (switchboard), Fax: +33 320574855

Shirong ZHAO
Corresponding Author. School of Finance, Dongbei University of Finance and Economics, Dalian 116025, Liaoning, China. shironz@163.com

IÉSEG School of Management Lille Catholic University 3, rue de la Digue F-59000 Lille Tel: 33(0)3 20 54 58 92
www.ieseg.fr

Testing Cost Function Convexity under Nonconvex Variable Returns-to-Scale Technologies: Evidence from U.S. Banks

Xiaoqing Chen[*] Kristiaan Kerstens[†] Shirong Zhao[‡]

June 9, 2026

Abstract

In this work, we assess the similarities between two nonconvex variable returns-to-scale nonparametric technologies: the traditional free disposal hull technology, and an alternative technology defined as the intersection of non-increasing and non-decreasing returns-to-scale technologies. The latter technology is slightly larger than the traditional free disposal hull, and it has rarely if ever been employed in empirical work. We also explore the differences between the corresponding two cost functions: the cost function based on the intersection of non-increasing and non-decreasing returns-to-scale cost functions is also larger than the traditional nonconvex variable returns to scale cost function and represents a new contribution to the literature. We define ways to compute both technologies and cost functions. We explore the empirical similarities between these two technologies and cost functions as well as their convex counterparts. We check whether both nonconvex technologies and cost functions make any differences in testing for convexity using a secondary data set: we still reject convexity under all circumstances.

Keywords: Nonconvex Technologies; Intersection Frontier; Cost Functions; Technical Efficiency; Cost Efficiency; Allocative Efficiency

[*]School of Management Science and Engineering, Nanjing University of Information Science and Technology, Nanjing, 210044, China. 003694@nuist.edu.cn

[†]Univ. Lille, CNRS, IESEG School of Management, UMR 9221 - LEM - Lille Économie Management, F-59000 Lille, France. k.kerstens@ieseg.fr. Tel: +33 320545892 (switchboard), Fax: +33 320574855

[‡]Corresponding Author. School of Finance, Dongbei University of Finance and Economics, Dalian 116025, Liaoning, China. shironz@163.com

1 Introduction

The empirical characterization of production technologies and their associated value functions, including cost and revenue functions, plays a fundamental role in applied economic analysis. Such models are routinely employed to evaluate efficiency, productivity, and scale economies and to support managerial and regulatory decision-making. A common feature of most nonparametric, semi-parametric, and parametric specifications is the imposition of convexity on the production technology and the corresponding value functions.

The standard convex nonparametric approach for evaluating the performance of homogeneous decision-making units (DMUs), introduced by Charnes, Cooper, and Rhodes (1978) and extended by Banker, Charnes, and Cooper (1984), is known as Data Envelopment Analysis (DEA) and constructs a piecewise linear production frontier and measures relative efficiency through mathematical programming. Within this framework, technical efficiency (TE) reflects the ability to produce maximal outputs at current inputs or to generate current outputs at minimal inputs, while cost efficiency (CE) measures the extent to which production costs are minimized given output requirements and input prices. Allocative efficiency (AE), following the framework of Farrell (1957) and Färe, Grosskopf, and Lovell (1985), captures the deviation between the observed input mix given input prices and the optimal input mix given input prices. These efficiency measures are derived under the maintained assumption of convexity in production, a feature that remains central to most empirical efficiency analyses.

Numerous approaches have been developed to evaluate technical and cost efficiency, with convex models remaining the dominant framework in both theory and practice. In particular, DEA and related methodologies have been extensively applied in economics, management science, and operations research (see Camanho, Silva, Piran, and Lacerda (2024) for a review). However, the convexity assumption underlying these models is often questioned. As emphasized by Farrell (1959), production technologies may exhibit nonconvexities arising from input and output indivisibilities (Scarf (1986) and Scarf (1994)), economies of scale and specialization (Romer, 1990), externalities, and the aggregation of heterogeneous production techniques (Hung, Le Van, and Michel, 2009). The economic relevance of such nonconvexities is well established. For instance, Scarf (1994) associates economies of scale with the presence of significant nonconvex production factors, while Eaton and Lipsey (1997) identify lumpy capital goods as a fundamental source of technological nonconvexity. These considerations motivate the study of nonconvex production technologies and their associated value functions as a complement to conventional convex approaches.

In empirical studies, potential sources of nonconvexity are commonly ignored, and convexity is justified through the assumption of perfect time divisibility (Shephard (1970, p. 15)). This assumption implies that production activities can be continuously scaled within a given planning horizon. However, convexity may fail when production processes involve non-zero setup times. Despite this concern, most empirical analyses proceed under the implicit assumption that nonconvexities have a negligible effect on the estimation of production technologies and associated value functions. Evidence supporting this assumption remains limited, particularly with respect to cost function estimation. Consequently, the extent to which convex specifications approximate nonconvex technologies and their corresponding cost functions remains largely unexplored. This issue motivates a direct comparison of convex and nonconvex formulations. Throughout the paper, convex (nonconvex) cost functions denote cost functions associated with convex (nonconvex) production technologies.

Alongside convex nonparametric technologies, a nonconvex deterministic framework, namely the Free Disposal Hull (FDH) model, has been introduced by Deprins, Simar, and Tulkens (1984). By imposing strong disposability of inputs and outputs while relaxing convexity, FDH provides an alternative benchmark technology for measuring technical efficiency. The original model is subsequently extended by Kerstens and Vanden Eeckaut (1999), who incorporate alternative returns-to-scale assumptions, including non-increasing, non-decreasing, variable, and constant returns to scale. Parallel developments have focused on the computational treatment of nonconvex technologies. For example, Tulkens (1993) proposes a mixed-integer programming formulation combined with implicit enumeration procedures, whereas Agrell and Tind (2001) develop a linear-programming-based approach for nonconvex efficiency analysis. This framework is later generalized by Leleu (2006) to accommodate alternative returns-to-scale technologies and their associated cost functions. In addition, Briec, Kerstens, and Vanden Eeckaut (2004) establish implicit enumeration procedures for a broad class of nonconvex technologies and value functions under different returns-to-scale assumptions.

Under convexity, the standard variable returns-to-scale (VRS) technology can be equivalently characterized as the intersection of non-increasing returns-to-scale (NIRS) and non-decreasing returns-to-scale (NDRS) technologies. Consequently, the conventional VRS cost function coincides with the cost function derived from the intersection of the corresponding NIRS and NDRS cost functions. This equivalence, however, does not generally extend to the nonconvex case. As noted by Briec, Kerstens, and Vanden Eeckaut (2004, Remark 1), two distinct nonconvex VRS technologies can be defined: the conventional FDH-VRS technology and an alternative technology obtained as the intersection of nonconvex NIRS and NDRS technologies. The latter generates a production possibility set that strictly contains the former. An analogous distinction arises for cost functions. In addition to the conventional FDH-VRS cost function, one may define a cost function associated with the intersection technology. To our knowledge, this intersection-based nonconvex cost function has not been explicitly investigated in the literature. Since the corresponding technology dominates the conventional FDH-VRS technology, the resulting cost function is also larger. Despite these conceptual differences, existing studies have generally treated the two formulations as equivalent or have focused exclusively on the conventional FDH specification.

Although the conventional FDH technology has been widely employed in empirical efficiency analysis across a variety of sectors and applications, empirical studies based on the intersection FDH technology remain relatively scarce. Existing contributions have largely focused on methodological developments or a limited number of empirical applications, leaving the practical implications of this alternative nonconvex specification largely unexplored. The imbalance is even more pronounced in the context of cost analysis. While several studies have estimated FDH-based cost functions, empirical evidence on cost functions derived from the intersection FDH technology is, to our knowledge, unavailable. Consequently, little is known about the extent to which the choice between the conventional FDH technology and its intersection counterpart affects cost estimation, efficiency measurement, and related empirical conclusions.

Recent work by Li, Tsang, Lee, and He (2025) highlights an additional feature of the intersection FDH technology. Specifically, it is argued that this technology is uniquely consistent with an S-shaped production process characterized by increasing returns to scale, followed by constant returns to scale (CRS), and subsequently decreasing returns to scale. This structure is referred to as regular variable returns to scale. Regardless of the theoretical status of this concept within the broader axiomatic production literature, the argument provides an additional motivation for investigating the empirical properties of the intersection FDH technology and its associated cost function.

The literature has also proposed technologies that impose convexity only on selected dimensions of the production set, most notably input-convex technologies (Chang (1999) and Cherchye, De Rock, and Hennebel (2014)). These models can be viewed as particular cases of the selectively convex framework developed by Podinovski (2005). The present study does not consider such specifications. First, their empirical applications remain limited. Second, their economic interpretation is less straightforward in the context of time divisibility, which is commonly invoked to justify convexity. In particular, it is unclear how convexity can be justified for only a subset of production dimensions while being absent for others.

Against this background, we compare alternative convex and nonconvex technologies and their associated cost functions using a sample of U.S. banks. The paper makes three main contributions. First, we establish new theoretical relationships between technical efficiency measures under alternative returns-to-scale assumptions in both convex and nonconvex settings and examine their empirical implications. Second, we derive corresponding results for cost functions and investigate whether alternative nonconvex specifications affect empirical conclusions regarding convexity. To this end, we employ the nonparametric convexity test of Li (1996), complemented by the procedures proposed by Kneip, Simar, and Wilson (2016) and Simar and Wilson (2020a). Third, we analyze the sources of cost inefficiency by decomposing cost efficiency into its technical and allocative components under both convex and nonconvex technologies. Together, these analyses provide new evidence on the empirical relevance of alternative nonconvex specifications and on the robustness of convexity assumptions in efficiency and cost-function analysis.

These statistical tests for convexity of technology have been so far only occasionally applied in empirical studies. Statistical evidence rejecting convexity is documented for a variety of sectors: e.g., USA (Wilson, 2021) and Chinese banking sectors (Wilson and Zhao, 2023b), Swedish dairy farms

(Ang, Owusu-Sekyere, Alvåsen, and Hansson, 2026), high-performance computing sites (Apon, Ngo, Payne, and Wilson, 2015), USA municipalities (O'Loughlin and Wilson, 2021), Belgian train traffic control centers (Kerstens, Roets, Van de Woestyne, and Zhao, 2026), the UK independent school sector (Lopez-Torres, Johnes, Elliott, and Polo, 2021), a macro-economic cross-country analysis covering 143 countries (Wilson and Zhao, 2025), Swedish district courts (Chen, Kerstens, and Zhu, 2025), among others. To the best of our knowledge, convexity of the cost function has only been tested and rejected in Kerstens and Zhao (2025) and Kerstens, O'Donnell, Van de Woestyne, and Zhao (2026). In essence, this contribution wants to assess whether an enlarged nonconvex intersection technology and cost function makes any impact on the eventual rejection of convexity.

The remainder of the paper is organized as follows. Section 2 introduces the production technologies, cost functions, and efficiency decompositions employed in the analysis. Section 3 presents graphical and numerical illustrations of technical efficiency for single input and single output under convex and nonconvex variable returns-to-scale technologies, as well as graphical and numerical illustrations of single output and single cost under convex and nonconvex variable returns-to-scale technologies. Section 4 discusses a data set of U.S. banks employed in the empirical illustration. Section 5 illustrates all empirical results in banking. The final Section 6 offers the conclusions.

2 Technology and Cost Functions and the Role of Convexity

This section first introduces the definitions of technology and cost functions together with their dual relationship. It then presents the nonparametric specifications employed to estimate technical and cost efficiency. Finally, cost efficiency is decomposed to identify its main determinants.

2.1 Definitions and Duality of Technology and Cost Function

Let $x \in \mathbb{R}^m_+$ and $y \in \mathbb{R}^s_+$ denote an m-dimensional input vector and an s-dimensional output vector, respectively. The production possibility set or technology T is defined as: $T = \{(x, y) \in \mathbb{R}^m_+ \times \mathbb{R}^s_+ \mid x \text{ can produce at least } y\}$. An equivalent representation is provided by the input requirement set: $L(y) = \{x \in \mathbb{R}^m_+ : (x, y) \in T\}$, which contains all input combinations capable of producing the output vector y.

The technology set T is assumed to satisfy some combination of the following standard axioms:

(T.1) $(0, 0) \in T$ and, if $(x, y) \in T$ with $x = 0$, then $y = 0$.

(T.2) T is closed in $\mathbb{R}^m_+ \times \mathbb{R}^s_+$.

(T.3) If $(x, y) \in T$ and $(x', y') \in \mathbb{R}^m_+ \times \mathbb{R}^s_+$, then $(x', -y') \geq (x, -y)$ implies $(x', y') \in T$.

(T.4) $(x, y) \in T \Rightarrow \delta(x, y) \in T$ for $\delta \in \Gamma$, where:

(i) $\Gamma^{\text{CRS}} = \delta : \delta \geq 0$;

(ii) $\Gamma^{\text{NDRS}} = \delta : \delta \geq 1$;

(iii) $\Gamma^{\text{NIRS}} = \delta : 0 \leq \delta \leq 1$;

(iv) Γ^{VRS} when none of the above restrictions hold.

(T.5) T is convex, i.e., for all $(x_1, y_1), (x_2, y_2) \in T$ and $\alpha \in [0, 1]$, $\alpha(x_1, y_1) + (1 - \alpha)(x_2, y_2) \in T$.

These axioms have well-established economic interpretations (see, e.g., Färe (1988) and Hackman (2008)). Axiom (T.1) combines the possibility of inaction with the absence of free production. Axiom (T.2) ensures topological regularity. Axiom (T.3) imposes strong disposability, allowing inputs to increase and outputs to decrease freely. Axiom (T.4) governs returns to scale and distinguishes between constant (CRS), non-increasing (NIRS), non-decreasing (NDRS), and variable returns to scale (VRS). Finally, Axiom (T.5) permits convex combinations of observed production plans. Because the empirical analysis contrasts convex and nonconvex technologies, convexity plays a particularly important role.

While (T.1) and (T.2) are generally regarded as innocuous regularity conditions, the remaining assumptions are more contentious. Strong disposability rules out congestion effects, despite empirical evidence that violations of monotonicity are not uncommon (see Shephard (1974) and Sauer (2006)). Likewise, Scarf (1994) questions the empirical relevance of the CRS assumption. Convexity is equally debatable, as discussed in the Introduction.

Note that a technology T^{NC} failing convexity $(T.5)$ is always extendable to a convex technology T^C by its convex hull. The technology $T^C = \{\alpha(x_1, y_1)+(1-\alpha)(x_2, y_2) \mid \forall (x_1, y_1) \in T^{NC}, \forall (x_2, y_2) \in T^{NC}, \forall \alpha \in [0, 1]\}$ represents the smallest convex technology containing T^{NC}.

The input distance function associated with the input requirement set $L(y)$ is as defined as:

$$D_i(x, y \mid T) = \max\{\alpha \, : \, \alpha \geq 0, (x/\alpha, y) \in T\} = \max\{\alpha \, : \, \alpha \geq 0, x/\alpha \in L(y)\}. \tag{1}$$

The input distance function possesses several important properties (see, e.g., Färe (1988) and Hackman (2008)). First, $D_i(x, y \mid T) \geq 1$, with a value of unity indicating that production takes place on the efficient isoquant of $L(y)$. Second, the measure admits a direct cost interpretation, which provides the basis for its dual relationship with the cost function.

The distinction between convex and nonconvex technologies has important implications for efficiency measurement and related performance indicators. Briec, Kerstens, and Van de Woestyne (2022) provide a comprehensive review of the evidence regarding efficiency decomposition, productivity growth, and capacity utilisation. Likewise, empirical studies such as Geys and Moesen (2009), Pieralli, Ritter, and Odening (2015), and Wang, Song, and Cullinane (2003) document potentially substantial differences in estimated technical inefficiency when convexity assumptions are relaxed.

The dual representation of technology is provided by the cost function, which measures the minimum expenditure required to produce a given output vector y at input prices $w \in \mathbb{R}^m_+$:

$$C(y, w \mid T) = \min\{wx \,:\, (x, y) \in T\}. \tag{2}$$

Duality theory links the primal representation of production technology to its dual cost representation. Under cost-minimising behaviour, a technology can be recovered from the associated cost function, while the converse is also true. Following Hackman (2008, p. 114), the dual relationship between the input distance function and the cost function is given by:

$$D_i(x, y \mid T) = \min_w\{wx \,:\, C(y, w \mid T) \geq 1, x \in L(y)\}, \tag{3}$$

$$C(y, w \mid T) = \min_x\{wx \,:\, D_i(x, y \mid T) \geq 1, w > 0\}, \tag{4}$$

Accordingly, the cost function is obtained by minimizing expenditure over feasible input bundles, whereas the input distance function can be recovered through optimization with respect to input prices.

Importantly, duality requires only convexity of the input requirement set rather than convexity of the entire technology (see Färe (1988, p. 84) and Jacobsen (1970, p. 760, 768)). Consequently, conventional cost functions derived from technologies satisfying axiom (T.5) impose stronger structural restrictions than are strictly necessary for duality. Extending this perspective, Briec, Kerstens, and Vanden Eeckaut (2004) establish a local duality result for nonconvex technologies satisfying strong disposability and alternative scaling properties, thereby providing a theoretical foundation for nonconvex cost functions.

Whether convexity materially affects cost measurement remains an empirical question. Existing evidence suggests that the impact can be substantial. De Borger and Kerstens (1996) report a difference of approximately 22% between convex and nonconvex cost estimates. Similarly, Kerstens and Van de Woestyne (2021) find that nonconvex costs exceed convex costs by 21% to 38% on average. Depending on the year considered, Kerstens and Zhao (2025) estimate convex costs to be 12%-17% lower than nonconvex costs, while Kerstens, O'Donnell, Van de Woestyne, and Zhao (2026) document an average gap of approximately 29%. Taken together, these findings indicate that convexity assumptions can have economically significant consequences for cost estimation and efficiency analysis.

2.2 Technology and Cost Functions: Nonparametric Specifications

Nonparametric production theory has been developed primarily within a convex framework (see, e.g., Afriat (1972) and Diewert and Parkan (1983)). The present study instead focuses on a family of technologies and cost functions that explicitly accommodates nonconvex production possibilities.

The point of departure is the free disposal hull (FDH) technology, which imposes free disposability of inputs and outputs without requiring convexity. Originally proposed for the single-output case by Afriat (1972), the FDH framework is subsequently extended to incorporate alternative returns-to-scale assumptions (Kerstens and Vanden Eeckaut, 1999) and later generalized to encompass nonconvex cost functions (Briec, Kerstens, and Vanden Eeckaut, 2004). Additional developments in nonconvex nonparametric production modelling are surveyed by Hackman (2008). Unless otherwise stated, the subsequent exposition follows Briec, Kerstens, and Vanden Eeckaut (2004).

A convenient way to represent both convex and nonconvex technologies under alternative returns-to-scale assumptions is through the unified formulation:

$$T^{\Lambda,\Gamma} = \left\{ (x,y) \,:\, x \geq \sum_{k=1}^{K} x_k \delta z_k, y \leq \sum_{k=1}^{K} y_k \delta z_k, z \in \Lambda, \delta \in \Gamma \right\}, \tag{5}$$

The set Γ determines the returns-to-scale structure, whereas Λ governs the aggregation of observed production activities. The scaling parameter δ controls the admissible expansion or contraction of observations and thereby captures CRS, NIRS, NDRS, or VRS technologies. The activity vector z determines whether observed production plans may be combined. When z is continuous and constrained to sum to unity, convex combinations of observations are feasible. When z is restricted to binary values, only observed activities themselves may be selected, yielding a nonconvex technology.

This formulation clearly separates the roles of convexity and returns to scale. Since the binary restriction is more restrictive than its continuous counterpart, every convex technology contains its nonconvex analogue: $T^{C,\Gamma} \supseteq T^{NC,\Gamma}$. Besides its analytical convenience, the unified representation provides a transparent framework for examining how convexity and scale assumptions independently affect efficiency and cost measurement.

The radial measure of input efficiency is given by the reciprocal of the input distance function: $E_i(x, y \mid T) = [D_i(x, y \mid T)]^{-1}$. Under the technology in model (5), radial input efficiency can be defined as:

$$E_i^{\Lambda,\Gamma}\left(x, y \mid T^{\Lambda,\Gamma}\right) = \min\left\{\alpha \;\mid\; (\alpha\, x, y) \in T^{\Lambda,\Gamma}, \alpha \geq 0\right\}. \tag{6}$$

For an observation $(x, y) \in T$, let $\alpha^* = E_i(x, y \mid T)$, then $(\alpha^* x, y) \in T$ denotes the radial input projection of (x, y) onto the boundary of the technology set.

Proposition 2.1. *(i)* $T^{\Lambda,CRS} = T^{\Lambda,NIRS} \cup T^{\Lambda,NDRS}$.
(ii) $T^{C,VRS} = T^{C,NIRS} \cap T^{C,NDRS}$.
(iii) $T^{NC,VRS} \subseteq T^{NC,\cap} = T^{NC,NIRS} \cap T^{NC,NDRS}$.

Proposition 2.1 characterizes the relationships among technologies constructed under alternative returns-to-scale assumptions. Under both convex and nonconvex formulations, CRS technologies emerge as the union of the NIRS and NDRS technologies. In the convex case, VRS technology is uniquely obtained as their intersection. By contrast, under nonconvexity the conventional FDH technology is strictly contained within the intersection of the NIRS and NDRS technologies.

Following Remark 1 of Briec, Kerstens, and Vanden Eeckaut (2004), the distinction is economically meaningful because the FDH-VRS technology satisfies the minimal extrapolation principle, while the intersection technology does not.

Computationally, evaluating the input distance function under convex technologies reduces to solving a linear programming problem equivalent to the standard DEA formulation (e.g., Färe (1988) and Hackman (2008)). For nonconvex technologies, the binary activity variables generate mixed-integer nonlinear optimization problems. Efficient implicit enumeration procedures are nevertheless available (Briec, Kerstens, and Vanden Eeckaut, 2004).

The unified formulation (5) immediately implies Proposition 2.2, which establishes that input efficiency under the nonconvex intersection technology is determined by the larger of the efficiencies obtained under NIRS and NDRS. Combining this result with the nesting relations among technologies yields the ordering:

Proposition 2.2. *For a given output level $y \in \mathbb{R}^s_+$ and a given input level $x \in \mathbb{R}^m_+$ under nonconvexity, $E_i(x, y \mid T^{NC,\cap}) = \max\{E_i(x, y \mid T^{NC,NIRS}), E_i(x, y \mid T^{NC,NDRS})\}$.*

Proposition 2.3. $E_i(x, y \mid T^{C,VRS}) = E_i(x, y \mid T^{C,\cap}) \leq E_i(x, y \mid T^{NC,\cap}) \leq E_i(x, y \mid T^{NC,VRS})$.

Employing the unified algebraic representation of the technology $T^{\Lambda,\Gamma}$ in (5), we obtain the following representation of the cost function in (7):

$$C(y, w \mid T^{\Lambda,\Gamma}) = \min \left\{ wx \,:\, x \geq \sum_{k=1}^{K} x_k \delta z_k, y \leq \sum_{k=1}^{K} y_k \delta z_k, z \in \Lambda, \delta \in \Gamma \right\}. \tag{7}$$

Because convex technologies encompass their nonconvex counterparts, the associated convex cost functions are no greater than their nonconvex counterparts.: $C(y, w \mid T^{C,\Gamma}) \leq C(y, w \mid T^{NC,\Gamma})$. Thus, imposing convexity can only reduce, or leave unchanged, the minimum cost required to produce a given output vector. As shown by Briec, Kerstens, and Vanden Eeckaut (2004), equality occurs only under CRS and a single-output technology.

The implications of this result have received surprisingly little attention in empirical applications. While cost efficiency studies almost universally rely on convex technologies, the possibility that convexity may induce systematic downward bias in estimated costs has rarely been examined. The computation of convex cost functions again reduces to standard linear programming problems (e.g., Färe (1988) and Hackman (2008)). In the nonconvex setting, the corresponding optimization problems can be solved using the implicit enumeration procedures developed by Briec, Kerstens, and Vanden Eeckaut (2004).

The cost function equation (7) lead directly to the following results:

Proposition 2.4. *(i) For a given output vector $y \in \mathbb{R}^s_+$ and price vector $w \in \mathbb{R}^m_+$ under convexity, $C(y, w \mid T^{C,VRS}) = \max\{C(y, w \mid T^{C,NIRS}), C(y, w \mid T^{C,NDRS})\}$.*
(ii) For a given output vector $y \in \mathbb{R}^s_+$ and price vector $w \in \mathbb{R}^m_+$ under nonconvexity, $C(y, w \mid T^{NC,VRS}) \geq C(y, w \mid T^{NC,\cap}) = \max\{C(y, w \mid T^{NC,NIRS}), C(y, w \mid T^{NC,NDRS})\}$.

Proof. The convex world has a unique VRS technology, following directly from the fact that $T^{C,\text{VRS}} = T^{C,\cap} = T^{C,\text{NIRS}} \cap T^{C,\text{NDRS}}$. However, the nonconvex world has two VRS technologies: FDH and FDH intersection, that is $T^{NC,\text{VRS}} \subseteq T^{NC,\cap} = T^{NC,\text{NIRS}} \cap T^{NC,\text{NDRS}}$. □

Proposition 2.5. *$C(y, w \mid T^{C,VRS}) \leq C(y, w \mid T^{NC,\cap}) \leq C(y, w \mid T^{NC,VRS})$.*

Recalling Proposition 2.1 of Briec, Kerstens, and Vanden Eeckaut (2004), costs evaluated relative to nonconvex technologies are always at least as large as those obtained under convex technologies, with equality arising only under CRS and a single output. Consequently, if the true technology is nonconvex, a convex cost function produces a downward-biased estimate of minimum cost. Conversely, when the underlying technology is convex, a nonconvex estimator remains consistent and converges asymptotically to the convex cost function, albeit at a slower convergence rate.[1]

2.3 Mahler Inequality and the Efficiency Decomposition of the Cost Function

For any input bundle $x \in L(y)$, the definition of the cost function in (7) implies:

$$C(y, w \mid T^{\Lambda,\Gamma}) \leq w \frac{x}{D_i(x, y \mid T^{\Lambda,\Gamma})} \tag{8}$$

The right-hand side represents the expenditure associated with the radial projection of the observed input vector onto the frontier. Thus, once technical inefficiency has been eliminated through a radial input contraction, the resulting expenditure provides an upper bound on minimum cost.

Rearranging model (8) yields Mahlers inequality (see, e.g., Färe and Grosskopf (2000)):

$$\frac{C(y, w \mid T^{\Lambda,\Gamma})}{wx} \leq \frac{1}{D_i(x, y \mid T^{\Lambda,\Gamma})} \tag{9}$$

The left-hand side defines cost efficiency as the ratio of the minimum feasible cost to the observed cost, whereas the right-hand side corresponds to the radial input efficiency measure $E_i^{\Lambda,\Gamma}\left(x, y \mid T^{\Lambda,\Gamma}\right)$).

Hence, cost efficiency cannot exceed radial technical efficiency. Any remaining discrepancy reflects the inability of the producer to select the cost-minimizing input mix.

[1] Simar and Wilson (2000, p. 56) establish these results for technology estimators; their extension to cost functions is immediate.

Introducing an allocative efficiency component transforms Mahler's inequality into an equality and yields the well-known multiplicative decomposition originally proposed by Farrell (1957). Farrell's framework constitutes the foundation of modern frontier efficiency analysis by distinguishing technical inefficiency from allocative inefficiency. Subsequent contributions, including Seitz (1971) and Färe, Grosskopf, and Lovell (1985), extended this taxonomy and established the decomposition that has become standard in the productivity and efficiency literature. Following Färe, Grosskopf, and Lovell (1985, p. 3-5), we adopt this framework throughout the remainder of the paper.

The efficiency concepts can be defined directly from the radial efficiency measure under a VRS technology. The relevant input-oriented measures are expressed as follows:

(a) Technical Efficiency: $TE^{\Lambda}(x, y) = E_i^{\Lambda,VRS}(x, y)$.
(b) Cost Efficiency: $CE^{\Lambda}(x, y, w) = C^{\Lambda,VRS}(y, w)/(w \cdot x)$.
(c) Allocative Efficiency: $AE^{\Lambda}(x, y, w) = CE^{\Lambda}(x, y, w)/TE^{\Lambda}(x, y)$.

Cost efficiency ($CE^{\Lambda}(x, y, w)$) evaluates the extent to which observed expenditure exceeds the minimum expenditure required to produce the observed output vector. Technical efficiency ($TE^{\Lambda}(x, y)$) measures the ability of a producer to operate on the boundary of the VRS technology. Allocative efficiency ($AE^{\Lambda}(x, y, w)$) captures the remaining loss arising from the choice of a non-cost-minimizing input mix after technical inefficiency has been removed.

These measures satisfy the familiar Farrell decomposition:

$$CE^{\Lambda}(x, y, w) = AE^{\Lambda}(x, y, w) \cdot TE^{\Lambda}(x, y). \tag{10}$$

which provides a mutually exclusive decomposition of overall cost efficiency into allocative and technical efficiency.

To develop subsequent tests for convexity, it is useful to establish how these efficiency measures vary across convex and nonconvex technologies. Extending Lemma 3 of Briec, Kerstens, and Vanden Eeckaut (2004), the following result follows immediately from the technology and cost rankings derived above.

Proposition 2.6. *For all $(x, y) \in \mathbb{R}_+^m \times \mathbb{R}_+^s$, the relations between nonconvex and convex decomposition components are: (a) $TE^C(x, y) \leq TE^{NC,\cap}(x, y) \leq TE^{NC,VRS}(x, y)$; (b) $CE^C(x, y, w) \leq CE^{NC,\cap}(x, y, w) \leq CE^{NC,VRS}(x, y, w)$.*

Proof. (a) follows from Proposition 2.3. (b) follows from Proposition 2.5. □

Proposition 2.6 demonstrates that relaxing convexity weakly increases both technical and cost efficiency. Intuitively, nonconvex technologies enlarge the set of admissible production possibilities relative to their convex counterparts, thereby improving the attainable efficiency scores.

By contrast, no analogous ranking can be established for allocative efficiency. Since allocative efficiency equals the ratio of cost efficiency to technical efficiency, both cost efficiency (numerator) and technical efficiency (denominator) rise when convexity is relaxed. Consequently, their relative magnitudes determine the resulting allocative efficiency score, and no unambiguous ordering exists between convex and nonconvex allocative efficiency measures. Although technical and cost efficiency can be ranked according to the convexity assumption, the same conclusion does not extend to allocative efficiency.

3 Graphical and Numerical Illustration

This section provides graphical and numerical illustrations of the nonconvex efficiency and cost models developed above. The objective is twofold. First, the graphical analysis helps visualize the theoretical distinctions between the alternative nonconvex technologies. Second, the numerical examples provide intuition for the efficiency and cost rankings established in the previous section and motivate the subsequent empirical application.

3.1 Graphical Illustration

Figure 1 depicts the efficiency frontiers generated by three nonconvex technologies together with the conventional FDH technology in a single-input, single-output setting. The solid yellow line represents the traditional FDH frontier ($T^{NC,VRS}$), while the blue and green dashed lines correspond to the FDH technologies satisfying non-increasing and non-decreasing returns to scale, respectively. The solid black line represents the frontier associated with the intersection technology $T^{NC,\cap}$.

Figure 1 provides a graphical illustration of Proposition 2.1. In particular, the intersection technology $T^{NC,\cap}$ is not identical to the conventional FDH technology $T^{NC,VRS}$. Rather, the former constitutes a larger technology set and therefore generates a frontier that differs from the traditional FDH frontier over part of the input-output space.

The distinction is illustrated by observations d and f. For observation d, the radial projections onto the two technologies coincide, implying identical efficiency scores. For observation f, however, the projection points differ, leading to different efficiency evaluations. This example demonstrates that the conventional FDH technology cannot generally be recovered as the intersection of the FDH technologies satisfying non-increasing and non-decreasing returns to scale.

To further illustrate the implications for cost measurement, Figure 2 presents the corresponding nonconvex cost frontiers in the single-output case. Observations a, b, c, d, and e are cost efficient under FDH. The staircase-shaped curve connecting these observations forms the conventional FDH cost frontier. As before, the solid yellow line denotes the traditional FDH cost frontier ($C^{NC,VRS}$),

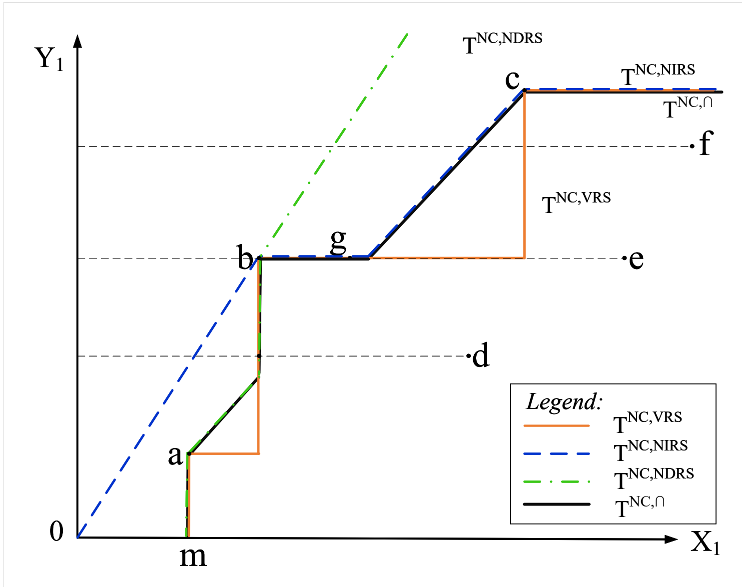

Figure 1: Illustrative diagram of efficiency frontiers under nonconvexity

the blue and green dashed lines correspond to the non-increasing and non-decreasing returns-to-scale cost frontiers, and the solid black line represents the cost frontier associated with the intersection technology ($C^{NC,\cap}$).

Figure 2 highlights the cost implications of the technological differences identified above. Consistent with Proposition 2.4, the cost frontier associated with the intersection technology differs from the conventional FDH cost frontier. Consequently, the traditional FDH cost model cannot generally be interpreted as the cost function obtained at the intersection of the non-increasing and non-decreasing returns-to-scale technologies.

This distinction is illustrated by observation 1. Under the conventional FDH technology, the observation projects onto point $1'$, whereas under the intersection technology it projects onto point $1''$. Since the two projection points correspond to different minimum-cost input bundles, the resulting cost efficiency measures also differ. The example therefore illustrates how seemingly small differences in the underlying technology specification can translate into economically meaningful differences in measured cost performance.

Taken together, Figures 1 and 2 provide graphical support for the theoretical results established in Section 2. They show that the intersection technology occupies an intermediate position between the traditional FDH technology and its scale-restricted counterparts, and that these technological differences carry over directly to efficiency and cost measurement.

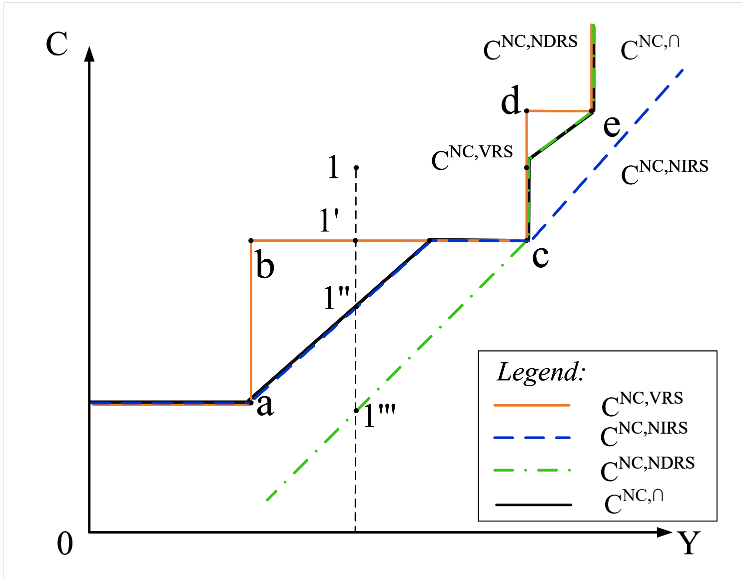

Figure 2: Illustrative diagram of cost frontiers under nonconvexity

3.2 Numerical Illustration

Assume that there is a single input and single output case with ten observations, we calculate the traditional convex and nonconvex inefficiencies, as well as their corresponding intersection frontiers based on the intersection of non-increasing and non-decreasing frontiers. The numerical data and efficiency results are shown in Table 1.

Table 1: Numerical Illustration of Technical Efficiency for One Input and One Output Case

DMUs	Indicators		Nonconvex		Convex	Δw.r.t. NC (%)	
	Input	Output	$TE^{NC,VRS}$	$TE^{NC,\cap}$	$TE^{C,VRS}$	TE^{VRS}	$TE^{\cap}$
1	8	228	1.0000	0.5700	0.4780	52.20	16.14
2	14	189	0.5714	0.2700	0.2437	57.35	9.74
3	3	150	1.0000	1.0000	1.0000	0.00	0.00
4	8	274	1.0000	0.6850	0.5387	46.13	21.35
5	8	164	1.0000	0.4100	0.3935	60.65	4.03
6	11	187	0.7273	0.3400	0.3083	57.61	9.34
7	9	718	1.0000	1.0000	1.0000	0.00	0.00
8	13	167	0.6154	0.2569	0.2446	60.26	4.80
9	18	341	0.5000	0.3789	0.2788	44.25	26.43
10	12	983	1.0000	1.0000	1.0000	0.00	0.00

Since the traditional convex efficiency coincides with the intersection result, we only list the traditional convex efficiency. From Table 1, observations 3, 7 and 10 are technically efficient for variable returns to scale and intersection specifications both under convexity and nonconvexity. Moreover, some observations are technically efficient under traditional nonconvexity ($TE^{NC,VRS}$), while technically inefficient under nonconvex intersection and traditional convex specifications, such as observations 1, 4 and 5. Last but not least, as we mentioned before, from the differences in terms of nonconvex technical estimates, the technical efficiency under TE^{VRS} measure is always no smaller than that under $TE^{NC,\cap}$ measure, and also equal to or larger than the technical efficiency obtained from $TE^{C,VRS}$ measure. Hence, part (iii) of Proposition 2.1 is confirmed by our findings.

In Table 2, we add the input price factor in the third column for the one input (second column) and one output (fourth column) case that is identical to Table 1. The cost efficiencies for traditional nonconvex and convex variable returns to scale specifications, and the intersection of non-increasing and non-decreasing returns to scale specifications have been calculated and reported in Table 2.

Table 2: Numerical Illustration of Cost Efficiency for One Input and One Output Case

DMUs	Indicators			Nonconvex		Convex	Δ w.r.t. NC (%)	
	Input	Price of Input	Output	$CE^{NC,VRS}$	$CE^{NC,\cap}$	$CE^{C,VRS}$	CE^{VRS}	$CE^{\cap}$
1	8	19.8	228	1.0000	0.5700	0.4780	52.20	16.14
2	14	27.2	189	0.5714	0.2700	0.2437	57.35	9.74
3	3	26.0	150	1.0000	1.0000	1.0000	0.00	0.00
4	8	24.0	274	1.0000	0.6850	0.5387	46.13	21.35
5	8	33.5	164	1.0000	0.4100	0.3935	60.65	4.03
6	11	22.1	187	0.7273	0.3400	0.3083	57.61	9.34
7	9	21.9	718	1.0000	1.0000	1.0000	0.00	0.00
8	13	36.2	167	0.6154	0.2567	0.2446	60.26	4.80
9	18	21.5	341	0.5000	0.3789	0.2788	44.25	26.43
10	12	34.8	983	1.0000	1.0000	1.0000	0.00	0.00

As shown in Table 2, our findings indicate that calculating cost efficiency for a single input is independent of the input price. The table presents cost efficiency estimates for 10 DMUs under three technological specifications: nonconvex VRS, nonconvex intersection frontier, and convex VRS. A key observation is the consistent full efficiency of DMUs 3, 7 and 10 across all specifications, which aligns with their technical efficiency results in Table 1 and confirms their optimal resource utilization and cost control. Notably, the results under the nonconvex intersection frontier strike a balance between the nonconvex VRS and convex VRS frontiers, with its mean cost efficiency being slightly lower than that of nonconvex VRS but significantly higher than that of the convex framework. This aligns with (ii) in Proposition 2.4, which states that $C(y, w \mid T^{NC,VRS}) \geq C(y, w \mid T^{NC,\cap}) = \max\{C(y, w \mid T^{NC,NIRS}), C(y, w \mid T^{NC,NDRS})\}$ under nonconvexity, and Proposition 2.5 ($C(y, w \mid T^{C,VRS}) \leq C(y, w \mid T^{NC,\cap}) \leq C(y, w \mid T^{NC,VRS})$).

Table 2 highlights three critical insights. First, technical efficiency is a necessary but not sufficient condition for cost efficiency in the single-input single-output case. While all technically efficient DMUs are cost-efficient, some technically inefficient ones (e.g., DMU 9) remain cost-inefficient due to persistent resource waste. Second, the nonconvex intersection frontier ($CE^{NC,\cap}$) exhibits intermediate stringency. For instance, its cost efficiency estimate for DMU 4 (0.6850) is lower than $CE^{NC,VRS}$ (1.0000) but higher than $CE^{C,VRS}$ (0.5210), which can be attributed to its definition as the intersection of non-increasing and non-decreasing returns to scale technologies. Third, convex VRS systematically underestimates cost efficiency, underscoring the risk of misclassifying observations as inefficient when convexity assumptions are imposed on inherently nonconvex production processes, such as those in the U.S. banking sector analyzed in this study.

4 Description of the Secondary Sample

The empirical analysis uses the banking dataset originally examined by Aly, Grabowski, Pasurka, and Rangan (1990). The sample consists of 322 independent commercial banks drawn from the Federal Deposit Insurance Corporation (FDIC) Reports of Condition and Reports of Income (Call Reports) for fiscal year 1986. Following Aly, Grabowski, Pasurka, and Rangan (1990), a nonparametric production framework based on the user-cost approach is adopted to define banking inputs, input prices, and outputs.

Three inputs are considered: labor (X_1), capital (X_2), and loanable funds (X_3). Labor is measured by the number of full-time employees on the payroll at the end of the reporting period. Capital is measured by the book value of premises and fixed assets, including capitalized leases. Loanable funds comprise time and savings deposits, notes and debentures, and other borrowed funds. Capital and loanable funds are expressed in thousands of dollars. Input prices are constructed in the standard user-cost fashion. The price of labor (P_1) is measured as total personnel expenditures divided by the number of employees. The price of capital (P_2) is computed as expenditures on premises and fixed assets divided by the corresponding book value of capital. Finally, the price of loanable funds (P_3) is obtained by dividing total interest expenses on deposits and borrowed funds by the stock of loanable funds. The production process is characterized by five outputs, all measured in thousands of dollars: real estate loans (Y_1), commercial and industrial loans (Y_2), consumer loans (Y_3), other loans (Y_4), and demand deposits (Y_5).

Table 3: Sample Description of Inputs and Outputs

Indicators	Mean	Stand. Dev	Min	Max
Labor (X_1)	32.043	60.704	3.000	816.000
Capital (X_2)	889.115	1766.448	2.000	16679.000
Loanable funds (X_3)	35680.975	61794.633	1186.000	830432.000
Price of Labor (P_1)	23.695	6.035	4.067	46.438
Price of Capital (P_2)	0.426	0.529	0.010	5.667
Price of Loanable funds (P_3)	0.065	0.011	0.003	0.086
Real estate loans (Y_1)	10769.957	18783.541	50.000	154801.000
Commercial and industrial loans (Y_2)	5984.183	17137.649	0.000	209909.000
Consumer loans (Y_3)	5530.286	16708.380	52.000	283888.000
Other loans (Y_4)	2631.823	5960.201	0.000	63536.000
Demand deposits (Y_5)	7700.127	17292.149	0.000	216784.000

Following Färe, Grosskopf, and Lovell (1994, p.44-45), the data must satisfy several regularity conditions for nonparametric production analysis. First, non-negative outputs require non-negative input quantities. Second, each input and output must be observed in positive amounts for at least some producers in the sample. Third, the production of a positive quantity of at least one output requires the use of a positive quantity of at least one input. The present dataset satisfies these requirements.

Table 3 reports descriptive statistics for all inputs, input prices, and outputs. Considerable heterogeneity is observed across banks, as reflected by the large standard deviations relative to the sample means. This variation is particularly pronounced for capital, loanable funds, and the major loan categories, suggesting substantial differences in scale among the sampled institutions.

Several outputs exhibit minimum values equal to zero. Such observations do not pose difficulties for the nonparametric analysis because banks may specialize in particular lending activities while continuing to produce other outputs. In other words, zero values for individual outputs remain compatible with an active production process and therefore satisfy the technological assumptions underlying the model.

Finally, the descriptive statistics reveal an inverse relationship between input quantities and input prices. The two largest input categoriescapital and loanable fundsdisplay substantially lower unit prices than labor. This pattern is consistent with the presence of scale-related cost advantages and provides an initial indication that scale considerations may play an important role in explaining efficiency and cost performance across banks.

5 Empirical Illustration

Our empirical analysis proceeds in three stages. We first compare technical efficiency estimates obtained under convex and nonconvex technologies. We then examine the corresponding cost efficiency measures. Finally, we decompose cost efficiency into its technical and allocative components to identify the principal sources of inefficiency.

5.1 Technical Efficiency for VRS under Convexity and Nonconvexity

We begin by comparing input-oriented technical efficiency scores obtained under three alternative technologies: the conventional convex VRS technology ($TE^{C,VRS}$), the traditional nonconvex FDH technology ($TE^{NC,VRS}$), and the nonconvex intersection technology ($TE^{NC,\cap}$). Table 4 reports descriptive statistics for these efficiency measures together with a series of statistical tests evaluating the convexity assumption. Specifically, the upper panel of Table 4 summarizes the distribution of efficiency scores. For each specification, we report the number of efficient observations, the mean efficiency score, the standard deviation, and the minimum and maximum values. The final two columns quantify the percentage difference between the convex and nonconvex estimates, calculated as $(TE^{NC} - TE^{C})/TE^{NC}$. In addition, the lower panel of Table 4 reports the results of three statistical procedures. We employ the Li-test and the adapted Li-test of Simar and Zelenyuk (2006) to compare the distributions of efficiency estimates, together with the convexity tests proposed by Kneip, Simar, and Wilson (2016) and Simar and Wilson (2020a).[2]

Table 4: Technical Efficiency for VRS under Convexity and Nonconvexity

Statistics	Nonconvex		Convex	Δw.r.t. NC (%)	
	$TE^{NC,VRS}$	$TE^{NC,\cap}$	$TE^{C,VRS}$	TE^{VRS}	$TE^{\cap}$
#Eff. Obs.	314	289	90	71.34	68.86
Mean	0.9957	0.9857	0.8021	19.44	18.63
Stand. Dev	0.0304	0.0524	0.1785	-487.84	-240.90
Min	0.6876	0.6196	0.3161	54.03	48.98
Max	1.0000	1.0000	1.0000	0.0000	0.0000

	$TE^{NC,VRS}$ vs. $TE^{C,VRS}$	$TE^{NC,\cap}$ vs. $TE^{C,VRS}$	$TE^{NC,VRS}$ vs. $TE^{NC,\cap}$
Li-test	146.5490***	126.1435***	-4.1617***
p-values	0.000	0.000	0.000
Adapted Li-test	117.6906***	—	—
p-values	0.000	—	—
KSW#1	2.7114***	—	—
p-values	0.000	—	—
KSW#2	0.8457***	—	—
p-values	0.000	—	—

Several findings emerge from Table 4. First, the number of efficient observations differs substantially across technologies. Under the convex VRS specification, only 90 of the 322 banks are identified as technically efficient. By contrast, the traditional FDH model classifies 314 banks as efficient, while the nonconvex intersection technology identifies 289 efficient observations. Thus, the number of efficient units is more than three times larger under the nonconvex specifications than under the convex benchmark. This result is consistent with Proposition 2.3: relaxing convexity enlarges the feasible production set and therefore increases the likelihood that an observation lies on the estimated frontier.

An additional insight concerns the distinction between the two nonconvex VRS technologies. Under convexity, the VRS and intersection technologies coincide, implying an identical number of efficient observations. Under nonconvexity, however, the two technologies differ, resulting in different frontier structures and consequently different classifications of efficient producers. This finding provides empirical support for the theoretical result that the traditional FDH technology is not equivalent to the intersection of the NIRS and NDRS technologies.

Second, substantial differences are also observed in the efficiency scores themselves. The average technical efficiency equals 0.9957 under the traditional FDH specification, 0.9857 under the nonconvex intersection technology, and only 0.8021 under the convex VRS model. Relative to the

[2] See Appendix A for technical details on this adapted Li-test.

nonconvex estimates, the convex efficiency scores are approximately 19.44% lower in the VRS comparison and 18.63% lower in the comparison with the intersection technology.

These magnitudes are economically significant. On average, the convex specification suggests that banks could reduce inputs by nearly 20% while maintaining current output levels, whereas the nonconvex technologies indicate that most banks operate very close to the efficient frontier. Moreover, the ranking of the average efficiency scores: $TE^{C,VRS} < TE^{NC,\cap} < TE^{NC,VRS}$, exactly mirrors the theoretical ordering established in Proposition 2.3. The empirical evidence therefore confirms that the choice of convexity assumption has a substantial impact on measured technical performance.

Third, the statistical tests strongly reject the hypothesis that convex and nonconvex efficiency measures are generated by the same underlying distribution. The Li-test statistics indicate highly significant differences between the distributions of $TE^{C,VRS}$ and $TE^{NC,VRS}$, as well as between $TE^{C,VRS}$ and $TE^{NC,\cap}$. The comparison between the two nonconvex measures is likewise statistically significant. The adapted Li-test proposed by Simar and Zelenyuk (2006) further confirms that the convex and nonconvex efficiency estimates differ substantially. In addition, both convexity tests of Kneip, Simar, and Wilson (2016) and Simar and Wilson (2020a) reject the null hypothesis of a convex technology at conventional significance levels. These findings are consistent with previous empirical evidence from the banking sector. For example, Wilson (2021) and Wilson and Zhao (2023a) likewise reject convexity when analyzing U.S. and Chinese banking technologies.

At present, the asymptotic properties of efficiency estimators based on the nonconvex intersection technology remain largely unexplored. In particular, the convergence rates and sampling distributions required to extend the tests of Simar and Zelenyuk (2006) and Simar and Wilson (2020a) to the intersection estimator have not yet been established. Consequently, no formal convexity test can currently be constructed using the intersection technology. Developing such procedures constitutes a promising avenue for future research.

Nevertheless, the available evidence points in a consistent direction. Both the distributional tests and the descriptive statistics indicate that the convexity assumption materially affects efficiency measurement in this sample of banks. Furthermore, significant differences are observed not only between convex and nonconvex technologies but also between the traditional FDH frontier and the nonconvex intersection frontier. These results reinforce the theoretical conclusion that alternative representations of nonconvex technologies can lead to economically meaningful differences in measured technical efficiency.

5.2 Cost Efficiency for VRS under Convexity and Nonconvexity

We next compare cost efficiency estimates under convex and nonconvex VRS cost frontiers. Table 5 reports the corresponding descriptive statistics. To facilitate comparison with the efficiency results presented earlier, Table 5 adopts the same layout and reporting structure as Table 4.

Table 5: Cost Efficiency for VRS under Convexity and Nonconvexity

Statistics	Nonconvex		Convex	Δw.r.t. NC (%)	
	$CE^{NC,VRS}$	$CE^{NC,\cap}$	$CE^{C,VRS}$	CE^{VRS}	$CE^{\cap}$
#Eff. Obs.	275	214	38	86.18	82.24
Mean	0.9760	0.9379	0.7078	27.48	24.54
Stand. Dev	0.0761	0.1134	0.1906	-150.67	-68.13
Min	0.5137	0.4552	0.2688	47.67	40.95
Max	1.0000	1.0000	1.0000	0.00	0.00

	$CE^{NC,VRS}$ vs. $CE^{C,VRS}$	$CE^{NC,\cap}$ vs. $CE^{C,VRS}$	$CE^{NC,VRS}$ vs. $CE^{NC,\cap}$
Li-test	175.3690***	133.8565***	17.0501**
p-values	0.000	0.000	0.0140
Adapted Li-test	188.1015***	—	—
p-values	0.000	—	—
KSW#1	4.6171***	—	—
p-values	0.000	—	—
KSW#2	0.9985***	—	—
p-values	0.002	—	—

First, the distribution of efficient observations differs markedly across technologies. Under the convex VRS specification, only 38 of the 322 observations are identified as cost efficient, whereas the number increases to 275 under the traditional nonconvex VRS technology and to 214 under the nonconvex intersection technology. This substantial increase in the number of efficient units suggests that the convexity assumption imposes a considerably more restrictive benchmark than its nonconvex counterparts. Furthermore, while the convex framework generates a unique VRS frontier, the nonconvex framework gives rise to two distinct VRS frontiers, namely the traditional nonconvex frontier and the intersection frontier. The divergence between these frontiers is reflected in the different numbers of efficient observations identified by $CE^{NC,VRS}$ and $CE^{NC,\cap}$.

Second, the efficiency levels differ substantially across alternative technological assumptions. The average cost efficiency score equals 0.7078 under the convex VRS technology, compared with 0.9760 under the traditional nonconvex specification and 0.9379 under the intersection technology. Hence, imposing convexity leads to considerably lower efficiency estimates, with average scores being approximately 27.48% lower relative to $CE^{NC,VRS}$ and 24.54% lower relative to $CE^{NC,\cap}$. These results indicate that the convexity assumption systematically enlarges the estimated distance to the cost frontier and therefore tends to portray production units as substantially less cost efficient. In contrast, the intersection estimator produces efficiency scores that lie between those obtained from the convex and traditional nonconvex technologies, suggesting that it provides a more conservative benchmark than the conventional nonconvex estimator while remaining substantially less restrictive than the convex specification.

Third, the statistical evidence strongly supports the existence of nonconvexity in the underlying cost technology. The Li-test statistics in Table 5 reject the null hypothesis of equality between the efficiency distributions obtained under convex and nonconvex specifications at conventional significance levels. The magnitude of these test statistics indicates that the observed differences are not merely numerical but reflect fundamentally different representations of the production technology. This conclusion is further reinforced by the adapted Li-test and by both KSW tests proposed by Kerstens and Zhao (2025), all of which reject the convexity assumption. Taken together, these findings provide compelling evidence that the empirical cost frontier is characterized by nonconvexities and that imposing convexity may lead to substantial distortions in efficiency measurement.

An additional contribution of the analysis concerns the comparison between the two nonconvex estimators. The Li-test also rejects the equality of the distributions associated with $CE^{NC,VRS}$ and $CE^{NC,\cap}$, indicating that the choice of nonconvex estimator remains empirically consequential even after abandoning convexity. This result suggests that the intersection technology captures aspects of the production structure that are not reflected by the traditional nonconvex frontier and therefore constitutes a distinct benchmark for efficiency evaluation.

Finally, it should be noted that the theoretical properties of cost-efficiency estimators based on the nonconvex intersection technology remain largely unexplored. In particular, asymptotic results required for constructing formal convexity tests analogous to those developed by Simar and Zelenyuk (2006) and Kerstens and Zhao (2025) are not yet available. Consequently, although the Li-test confirms statistically significant differences between the convex frontier and the nonconvex intersection technology, a direct convexity test based on the latter estimator cannot currently be implemented. Developing such inferential procedures represents a promising direction for future methodological research.

5.3 Allocative Efficiency for VRS under Convexity and Nonconvexity

The comparison of allocative efficiency under convex and nonconvex VRS technologies is presented in Table 6. This empirical analysis reveals several important insights. First, the pattern of efficient observations closely mirrors that observed for cost efficiency. Under the convex VRS technology, only 38 observations are identified as allocatively efficient, whereas 275 observations are efficient under the traditional nonconvex VRS frontier and 214 under the nonconvex intersection frontier. As in the cost-efficiency analysis, the substantially larger number of efficient observations under nonconvexity indicates that convexity imposes a considerably more restrictive benchmark. Moreover, the discrepancy between $AE^{NC,VRS}$ and $AE^{NC,\cap}$ further confirms that two distinct VRS frontiers emerge under nonconvexity, whereas convexity yields a unique frontier representation.

Second, allocative efficiency levels also differ systematically across alternative technological specifications. The average allocative efficiency score equals 0.8819 under the convex technology, compared with 0.9797 under the traditional nonconvex frontier and 0.9500 under the intersection

Table 6: Allocative Efficiency for VRS under Convexity and Nonconvexity

Statistics	Nonconvex		Convex	Δw.r.t. NC (%)	
	$AE^{NC,VRS}$	$AE^{NC,\cap}$	$AE^{C,VRS}$	AE^{VRS}	$AE^{\cap}$
#Eff. Obs.	275	214	38	86.18	82.24
Mean	0.9797	0.9500	0.8819	9.98	7.17
Stand. Dev	0.0654	0.0926	0.1195	-82.80	-29.07
Min	0.5137	0.4552	0.3516	31.55	22.76
Max	1.0000	1.0000	1.0000	0.00	0.00

	$AE^{NC,VRS}$ vs. $AE^{C,VRS}$	$AE^{NC,\cap}$ vs. $AE^{C,VRS}$	$AE^{NC,VRS}$ vs. $AE^{NC,\cap}$
Li-test	167.3046***	125.5099***	17.0512***
p-values	0.000	0.000	0.0100
Adapted Li-test	146.6128***	—	—
p-values	0.000	—	—
KSW#1	1.5608**	—	—
p-values	0.043	—	—
KSW#2	0.5926*	—	—
p-values	0.086	—	—

frontier. Thus, convexity reduces average allocative efficiency estimates by approximately 9.98% relative to $AE^{NC,VRS}$ and by more than 7.17% relative to $AE^{NC,\cap}$. As observed for cost efficiency, the intersection estimator produces values that lie between the traditional nonconvex and convex benchmarks, suggesting that it constitutes an intermediate representation of the underlying production technology.

More importantly, combining the evidence from Tables 4-6 reveals a fundamental shift in the sources of cost inefficiency across technological assumptions. Under the nonconvex technologies, technical efficiency is on average close to unity, whereas allocative efficiency remains comparatively lower. This finding indicates that deviations from cost minimization primarily originate from suboptimal input allocation rather than from technological inefficiency. In contrast, under the convex technology, technical efficiency exhibits a substantially larger deterioration than allocative efficiency. Consequently, the dominant source of cost inefficiency switches from allocative inefficiency under nonconvexity to technical inefficiency under convexity. This result is particularly important because it implies that the diagnosis of managerial performance depends critically on the maintained technological assumption. In other words, imposing convexity not only affects the magnitude of efficiency estimates but also alters the interpretation of the underlying sources of inefficiency.

The statistical evidence strongly supports these conclusions. The Li-test statistics reject the equality of allocative-efficiency distributions obtained under convex and nonconvex technologies at conventional significance levels. Furthermore, both KSW tests reject the convexity assumption, providing additional evidence that allocative efficiency is evaluated under fundamentally different frontier structures when convexity is imposed. To the best of our knowledge, this constitutes one of the first empirical applications reporting a direct rejection of convexity for the allocative-efficiency component itself. The finding extends the growing body of evidence questioning the empirical validity of convex technologies in efficiency analysis.

The comparison between the two nonconvex estimators also yields statistically significant differences. The Li-test rejects the equality of the distributions associated with $AE^{NC,VRS}$ and $AE^{NC,\cap}$, indicating that the choice of nonconvex estimator remains empirically relevant even after abandoning the convexity assumption. The intersection frontier therefore should not be viewed merely as a technical refinement of the traditional nonconvex estimator; rather, it represents a distinct benchmark that generates systematically different allocative-efficiency assessments.

Finally, it should be acknowledged that the asymptotic properties of allocative-efficiency estimators derived from the nonconvex intersection technology remain largely unexplored. As a consequence, formal convexity tests analogous to those developed by Simar and Zelenyuk (2006) and Kerstens and Zhao (2025) cannot yet be constructed for this estimator. Developing such inferential procedures represents an important avenue for future methodological research.

Overall, the results reinforce the conclusion that the nonconvex intersection frontier occupies an intermediate position between the traditional nonconvex and convex VRS technologies. While preserving the flexibility of nonconvex approaches, it introduces additional discipline through the intersection construction and thereby generates efficiency estimates that are systematically bounded between the two extremes. More broadly, the findings demonstrate that technological assumptions influence not only the level of efficiency scores but also the identification of the underlying sources of inefficiency, highlighting the importance of carefully assessing the validity of convexity in empirical efficiency studies.

5.4 Sensitivity Analysis

In this section, we conduct a sensitivity analysis to assess the robustness of our empirical findings. From the full sample of 322 observations, we randomly draw two subsamples of sizes 300 and 200. For each subsample, we compute technical efficiency, cost efficiency, and allocative efficiency under both the convex technology and the two nonconvex technologies. We also perform the same set of tests as in Subsections 5.1 to 5.3.

The results are reported in Table 7 for the 300-observation subsample and Table 8 for the 200-observation subsample. The patterns observed in these tables are consistent with those in the full sample. As before, the nonconvex intersection frontier lies between the traditional nonconvex and the convex VRS technologies: while it retains the nonconvex technologies ability to classify more observations as efficient, its intersection-based construction imposes moderate additional stringency, yielding efficiency measures that fall between those of the two extremes. Furthermore, the Li-tests and KSW tests all show that cost, technical, and allocative efficiency differ significantly between the convex and traditional nonconvex technologies. The Li-test statistics also confirm significant

Table 7: Decomposition Effects under Convexity and Nonconvexity with 300 Observations

Statistics	Decomposition Effects of Cost Efficiency Estimates								
	Nonconvex			Convex			Δw.r.t. NC (%)		
	$CE^{NC,VRS}$	$TE^{NC,VRS}$	$AE^{NC,VRS}$	$CE^{C,VRS}$	$TE^{C,VRS}$	$AE^{C,VRS}$	$CE^{\cdot,VRS}$	$TE^{\cdot,VRS}$	$AE^{\cdot,VRS}$
#Eff. Obs.	253	292	253	31	84	31	87.75	71.23	87.75
Mean	0.9742	0.9953	0.9782	0.6974	0.7959	0.8767	28.41	20.03	10.38
Stand. Dev	0.0785	0.0314	0.0675	0.1886	0.1792	0.1220	-140.25	-470.70	-80.74
Min	0.5137	0.6876	0.5137	0.2688	0.3161	0.3516	47.67	54.03	31.56
Max	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.0000	0.0000	0.0000
	$CE^{NC,VRS}$ vs. $CE^{C,VRS}$			$TE^{NC,VRS}$ vs. $TE^{C,VRS}$			$AE^{NC,VRS}$ vs. $AE^{C,VRS}$		
Li-test	165.0514***			135.1057***			153.5584***		
p-values	0.000			0.000			0.000		
Adapted Li-test	174.468***			115.0166***			135.1580***		
p-values	0.000			0.000			0.000		
KSW#1	4.6556***			2.1911***			1.7347**		
p-values	0.001			0.000			0.030		
KSW#2	0.9752**			0.6694***			0.8278***		
p-values	0.152			0.002			0.004		
Statistics	Decomposition Effects of Cost Intersection Efficiency Estimates								
	Nonconvex			Convex			Δw.r.t. NC (%)		
	$CE^{NC,\cap}$	$TE^{NC,\cap}$	$AE^{NC,\cap}$	$CE^{C,VRS}$	$TE^{C,VRS}$	$AE^{C,VRS}$	$CE^{\cap}$	$TE^{\cap}$	$AE^{\cap}$
#Eff. Obs.	210	275	210	31	84	31	85.24	69.45	85.24
Mean	0.9439	0.9877	0.9543	0.6974	0.7959	0.8767	26.12	19.42	8.13
Stand. Dev	0.1110	0.0509	0.0914	0.1886	0.1792	0.1220	-69.91	-252.06	-33.48
Min	0.4552	0.6196	0.4552	0.2688	0.3161	0.3516	40.95	48.98	22.76
Max	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.0000	0.0000	0.0000
	$CE^{NC,\cap}$ vs. $CE^{C,VRS}$			$TE^{NC,\cap}$ vs. $TE^{C,VRS}$			$AE^{NC,\cap}$ vs. $AE^{C,VRS}$		
Li-test	135.2482***			115.8936***			126.1185***		
p-values	0.000			0.000			0.000		
	$CE^{NC,VRS}$ vs. $CE^{NC,\cap}$			$TE^{NC,VRS}$ vs. $TE^{NC,\cap}$			$AE^{NC,VRS}$ vs. $AE^{NC,\cap}$		
Li-test	14.0296			-9.1389***			18.3414		
p-values	0.2320			0.0050			0.6560		

differences between the convex technology and the intersection-based nonconvex technology. Therefore, the evidence presented in Tables 7-8 reinforces the conclusion that our empirical findings are robust.

6 Conclusions

We revisit the banking data originally analysed by Aly, Grabowski, Pasurka, and Rangan (1990), who employed the traditional input-oriented convex DEA framework to evaluate technical, allocative, scale, and cost efficiencies for a sample of 322 U.S. banks. Building upon this seminal dataset, we investigate how alternative assumptions regarding the shape of the production technology affect efficiency measurement. In particular, we compare conventional convex variable-returns-to-scale (VRS) frontiers with their nonconvex counterparts and with the intersection technology obtained from non-increasing and non-decreasing returns-to-scale specifications.

Table 8: Decomposition Effects under Convexity and Nonconvexity with 200 Observations

Statistics	Decomposition Effects of Cost Efficiency Estimates								
	Nonconvex			Convex			Δw.r.t. NC (%)		
	$CE^{NC,VRS}$	$TE^{NC,VRS}$	$AE^{NC,VRS}$	$CE^{C,VRS}$	$TE^{C,VRS}$	$AE^{C,VRS}$	$CE^{\cdot,VRS}$	$TE^{\cdot,VRS}$	$AE^{\cdot,VRS}$
#Eff. Obs.	176	196	176	39	80	39	77.84	59.18	77.84
Mean	0.9792	0.9960	0.9826	0.7292	0.8344	0.8702	25.53	16.22	11.44
Stand. Dev	0.0722	0.0292	0.0619	0.2022	0.1768	0.1268	-180.06	-505.48	-104.85
Min	0.5137	0.7285	0.5137	0.2688	0.3376	0.4548	47.67	53.66	11.47
Max	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.0000	0.0000	0.0000
	$CE^{NC,VRS}$ vs. $CE^{C,VRS}$			$TE^{NC,VRS}$ vs. $TE^{C,VRS}$			$AE^{NC,VRS}$ vs. $AE^{C,VRS}$		
Li-test	165.0514***			135.1057***			153.5584***		
p-values	0.000			0.000			0.000		
Adapted Li-test	174.4680***			115.0166***			135.158***		
p-values	0.000			0.000			0.000		
KSW#1	4.4331***			2.1612***			2.3655		
p-values	0.003			0.000			0.185		
KSW#2	0.9917*			0.6979***			0.8775*		
p-values	0.058			0.004			0.078		
Statistics	Decomposition Effects of Cost Intersection Efficiency Estimates								
	Nonconvex			Convex			Δw.r.t. NC (%)		
	$CE^{NC,\cap}$	$TE^{NC,\cap}$	$AE^{NC,\cap}$	$CE^{C,VRS}$	$TE^{C,VRS}$	$AE^{C,VRS}$	$CE^{\cap}$	$TE^{\cap}$	$AE^{\cap}$
#Eff. Obs.	149	185	149	39	80	39	73.83	56.76	73.83
Mean	0.9519	0.9902	0.9599	0.7292	0.8344	0.8702	23.40	15.73	9.34
Stand. Dev	0.1036	0.0429	0.0856	0.2022	0.1768	0.1268	-95.17	-312.12	-48.13
Min	0.4989	0.7285	0.5137	0.2688	0.3376	0.4548	46.12	53.66	11.47
Max	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.0000	0.0000	0.0000
	$CE^{NC,\cap}$ vs. $CE^{C,VRS}$			$TE^{NC,\cap}$ vs. $TE^{C,VRS}$			$AE^{NC,\cap}$ vs. $AE^{C,VRS}$		
Li-test	135.2482***			115.8936***			126.1185***		
p-values	0.000			0.000			0.000		
	$CE^{NC,VRS}$ vs. $CE^{NC,\cap}$			$TE^{NC,VRS}$ vs. $TE^{NC,\cap}$			$AE^{NC,VRS}$ vs. $AE^{NC,\cap}$		
Li-test	14.0296			-9.1389***			18.3414		
p-values	0.2320			0.0050			0.6560		

The analysis generates several important findings. First, the empirical evidence consistently indicates that convexity constitutes a highly restrictive assumption in the banking industry. Across technical, cost, and allocative efficiency measures, substantially more observations are identified as efficient under nonconvex technologies than under the conventional convex specification. Moreover, all statistical tests considered in the analysis reject the equality of efficiency distributions obtained under convex and nonconvex technologies, while the KSW tests provide direct evidence against the convexity assumption itself. Taken together, these findings suggest that the production technology underlying banking operations is more appropriately represented by a nonconvex frontier.

Second, the results reveal that abandoning convexity affects not only the magnitude of efficiency estimates but also their economic interpretation. Under convex technologies, cost inefficiency is primarily driven by technical inefficiency. In contrast, under nonconvex technologies, allocative inefficiency emerges as the dominant source of departures from cost minimization. This finding has important managerial implications because the diagnosis of performance shortcomings depends critically on the maintained technological specification. Consequently, managerial recommendations derived from convex and nonconvex models may differ substantially, even when based on the same underlying data.

Third, the analysis provides new evidence regarding the role of the intersection frontier under nonconvexity. The empirical results consistently position the nonconvex intersection technology between the traditional nonconvex and convex VRS frontiers. While preserving the flexibility of nonconvex technologies, the intersection approach introduces an additional degree of discipline that produces more conservative efficiency estimates than the conventional nonconvex benchmark. At the same time, the significant differences observed between the traditional nonconvex and intersection estimators indicate that the latter should be regarded as a distinct frontier representation rather than merely a refinement of existing nonconvex approaches.

The study also contributes to the methodological literature. To the best of our knowledge, this is the first empirical investigation that jointly compares convex VRS, traditional nonconvex VRS, and nonconvex intersection VRS technologies for the decomposition of cost efficiency into its technical and allocative components. The theoretical analysis demonstrates that convexity implies a unique VRS frontier, whereas nonconvexity gives rise to two distinct VRS frontier concepts. The empirical evidence further illustrates that these alternative frontier representations can generate substantially different efficiency assessments and managerial conclusions.

More broadly, the findings contribute to the growing literature questioning the routine imposition of convexity in efficiency analysis. Although convexity has long been treated as a standard assumption in banking studies, empirical support for this assumption is rarely examined formally. Our results, together with the evidence reported by Kerstens, O'Donnell, Van de Woestyne, and Zhao (2026), suggest that the validity of convexity should be viewed as an empirical question rather than a maintained assumption. Future studies may therefore benefit from systematically evaluating alternative technological specifications before drawing conclusions regarding firm performance.

Several limitations remain. In particular, the asymptotic properties of efficiency estimators based on the nonconvex intersection technology have not yet been fully established. As a result, formal convexity tests analogous to those proposed by Simar and Zelenyuk (2006) and Simar and Wilson (2020a) cannot currently be extended to the intersection estimator. Developing the statistical foundations of intersection-based efficiency measures therefore represents a promising avenue for future research. Such advances would facilitate more rigorous inference regarding technological assumptions and further enhance the practical usefulness of nonconvex frontier methods.

Declaration of competing interest

The authors confirm the absence of any financial or personal ties with individuals or organizations that could have biased the findings of this study. They also affirm that they hold no commercial or

associative interests that could create a conflict of interest regarding this work.

Acknowledgment

The authors are indebted to Pieter Jan Kerstens for his insightful and valuable comments on this project. This paper has been funded by the National Natural Science Foundation of China (nos. 72301145, 72401056).

References

Afriat, S. (1972): "Efficiency Estimation of Production Functions," *International Economic Review*, 13(3), 568–598.

Agrell, P. J., and J. Tind (2001): "A Dual Approach to Nonconvex Frontier Models," *Journal of Productivity Analysis*, 16(2), 129–147.

Aly, H. Y., R. Grabowski, C. Pasurka, and N. Rangan (1990): "Technical, Scale, and Allocative Efficiencies in US Banking: An Empirical Investigation," *Review of Economics and Statistics*, 72(2), 211–218.

Ang, F., E. Owusu-Sekyere, K. Alvåsen, and H. Hansson (2026): "Accounting for Animal Health in Efficiency Analysis: An Application to Swedish Dairy Farmss," *American Journal of Agricultural Economics*, forthcoming(forthcoming), forthcoming.

Apon, A., L. Ngo, M. Payne, and P. Wilson (2015): "Assessing the Effect of High Performance Computing Capabilities on Academic Research Output," *Empirical Economics*, 48(1), 283–312.

Banker, R. D., A. Charnes, and W. W. Cooper (1984): "Some Models for Estimating Technical and Scale Inefficiencies in Data Envelopment Analysis," *Management Science*, 30(9), 1078–1092.

Briec, W., K. Kerstens, and I. Van de Woestyne (2022): "Nonconvexity in Production and Cost Functions: An Exploratory and Selective Review," in *Handbook of Production Economics*, ed. by S. Ray, R. Chambers, and S. Kumbhakar, vol. 2, pp. 721–754. Springer, Singapore.

Briec, W., K. Kerstens, and P. Vanden Eeckaut (2004): "Non-convex Technologies and Cost Functions: Definitions, Duality and Nonparametric Tests of Convexity," *Journal of Economics*, 81(2), 155–192.

Camanho, A., M. Silva, F. Piran, and D. Lacerda (2024): "A Literature Review of Economic Efficiency Assessments Using Data Envelopment Analysis," *European Journal of Operational Research*, 315(1), 1–18.

Chang, K.-P. (1999): "Measuring Efficiency with Quasiconcave Production Frontiers," *European Journal of Operational Research*, 115(3), 497–506.

Charnes, A., W. W. Cooper, and E. Rhodes (1978): "Measuring the Efficiency of Decision Making Units," *European Journal of Operational Research*, 2(6), 429–444.

Chen, X., K. Kerstens, and Q. Zhu (2025): "Exploring Horizontal Mergers in Swedish District Courts using Technical and Scale Efficiency: Rejecting Convexity in Favour of Nonconvexity," *Annals of Operations Research*, 351(2), 1319–1351.

Cherchye, L., B. De Rock, and V. Hennebel (2014): "The Economic Meaning of Data Envelopment Analysis: A 'Behavioral' Perspective," *Socio-Economic Planning Sciences*, 48(1), 29–37.

De Borger, B., and K. Kerstens (1996): "Cost Efficiency of Belgian Local Governments: A Comparative Analysis of FDH, DEA, and Econometric Approaches," *Regional Science and Urban Economics*, 26(2), 145–170.

Deprins, D., L. Simar, and H. Tulkens (1984): "Measuring Labor Efficiency in Post Offices," in *The Performance of Public Enterprises: Concepts and Measurements*, ed. by M. Marchand, P. Pestieau, and H. Tulkens, pp. 243–268. North Holland, Amsterdam.

Diewert, W., and C. Parkan (1983): "Linear Programming Test of Regularity Conditions for Production Functions," in *Quantitative Studies on Production and Prices*, ed. by W. Eichhorn, K. Neumann, and R. Shephard, pp. 131–158. Physica-Verlag, Würzburg.

Eaton, B., and R. Lipsey (1997): *On the Foundations of Monopolistic Competition and Economic Geography: The Selected Essays of B. Curtis Eaton and Richard G. Lipsey*. Edward Elgar, Cheltenham.

Färe, R. (1988): *Fundamentals of Production Theory*. Springer, Berlin.

Färe, R., and S. Grosskopf (2000): "Notes on some Inequalities in Economics," *Economic Theory*, 15(1), 227–233.

Färe, R., S. Grosskopf, and C. Lovell (1994): *Production Frontiers*. Cambridge University Press, Cambridge.

Färe, R., S. Grosskopf, and C. K. Lovell (1985): *The Measurement of Efficiency of Production*, vol. 6. Springer Science & Business Media.

Farrell, M. (1957): "The Measurement of Productive Efficiency," *Journal of the Royal Statistical Society Series A: General*, 120(3), 253–281.

Farrell, M. (1959): "The Convexity Assumption in the Theory of Competitive Markets," *Journal of Political Economy*, 67(4), 377–391.

Geys, B., and W. Moesen (2009): "Measuring Local Government Technical (In)efficiency: An Application and Comparison of FDH, DEA and Econometric Approaches," *Public Performance & Management Review*, 32(4), 499–513.

Hackman, S. (2008): *Production Economics: Integrating the Microeconomic and Engineering Perspectives*. Springer, Berlin.

Hung, N., C. Le Van, and P. Michel (2009): "Non-convex Aggregate Technology and Optimal Economic Growth," *Economic Theory*, 40(3), 457–471.

Jacobsen, S. (1970): "Production Correspondences," *Econometrica*, 38(5), 754–771.

Kerstens, K., C. O'Donnell, I. Van de Woestyne, and S. Zhao (2026): "When Cost Metafrontiers are Nonconvex in the Outputs, then the Production Metafrontier is Nonconvex: The Price of a Convexification Strategy," *European Journal of Operational Research*, 328(1), 324–335.

Kerstens, K., B. Roets, I. Van de Woestyne, and S. Zhao (2026): "An Exploratory Analysis of Learning from Peers: Radial vs. Nonradial Efficiency Measures and Convex vs. Nonconvex Technologies," *European Journal of Operational Research*, 329(3), 1052–1062.

Kerstens, K., and I. Van de Woestyne (2021): "Cost Functions are Nonconvex in the Outputs when the Technology is Nonconvex: Convexification is Not Harmless," *Annals of Operations Research*, 305(1–2), 81–106.

Kerstens, K., and P. Vanden Eeckaut (1999): "Estimating Returns to Scale Using Nonparametric Deterministic Technologies: A New Method Based on Goodness-of-Fit," *European Journal of Operational Research*, 113(1), 206–214.

Kerstens, K., and S. Zhao (2025): "Testing the Convexity Hypothesis in Nonparametric Cost Functions," *Economics Letters*, 247, 112196.

Kneip, A., L. Simar, and P. W. Wilson (2016): "Testing Hypotheses in Nonparametric Models of Production," *Journal of Business and Economic Statistics*, 34(3), 435–456.

Leleu, H. (2006): "A Linear Programming Framework for Free Disposal Hull Technologies and Cost Functions: Primal and Dual Models," *European Journal of Operational Research*, 168(2), 340–344.

Li, Q. (1996): "Nonparametric Testing of Closeness between Two Unknown Distribution Functions," *Econometric Reviews*, 15(3), 261–274.

Li, Q., E. Maasoumi, and J. S. Racine (2009): "A Nonparametric Test for Equality of Distributions with Mixed Categorical and Continuous Data," *Journal of Econometrics*, 148(2), 186–200.

Li, S.-K., C. Tsang, S. Lee, and X. He (2025): “Regular Variable Returns to Scale Production Frontier and Efficiency Measurement,” *Operations Research*, 73(5), 2830–2848.

Lopez-Torres, L., J. Johnes, C. Elliott, and C. Polo (2021): “The Effects of Competition and Collaboration on Efficiency in the UK Independent School Sector,” *Economic Modelling*, 96, 40–53.

O’Loughlin, C., and P. Wilson (2021): “Benchmarking the Performance of US Municipalities,” *Empirical economics*, 60(6), 2665–2700.

Pieralli, S., M. Ritter, and M. Odening (2015): “Efficiency of Wind Power Production and Its Determinants,” *Energy*, 90, 429–438.

Podinovski, V. (2005): “Selective Convexity in DEA Models,” *European Journal of Operational Research*, 161(2), 552–563.

Romer, P. (1990): “Are Nonconvexities Important for Understanding Growth?,” *American Economic Review*, 80(2), 97–103.

Sauer, J. (2006): “Economic Theory and Econometric Practice: Parametric Efficiency Analysis,” *Empirical Economics*, 31(4), 1061–1087.

Scarf, H. (1986): “Testing for Optimality in the Absence of Convexity,” in *Social Choice and Public Decision Making: Essays in Honor of Kenneth J. Arrow, Volume I*, ed. by W. Heller, R. Starr, and S. Starrett, pp. 117–134. Cambridge University Press, Cambridge.

——— (1994): “The Allocation of Resources in the Presence of Indivisibilities,” *Journal of Economic Perspectives*, 8(4), 111–128.

Seitz, W. D. (1971): “Productive Efficiency in the Steam-Electric Generating Industry,” *Journal of Political Economy*, 79(4), 878–886.

Shephard, R. (1970): *Theory of Cost and Production Functions*. Princeton University Press, Princeton.

——— (1974): *Indirect Production Functions*. Verlag Anton Hain, Meisenheim am Glam.

Simar, L., and P. Wilson (2000): “Statistical Inference in Nonparametric Frontier Models: The State of the Art,” *Journal of Productivity Analysis*, 13(1), 49–78.

Simar, L., and P. W. Wilson (2020a): “Hypothesis Testing in Nonparametric Models of Production using Multiple Sample Splits,” *Journal of Productivity Analysis*, 53(3), 287–303.

——— (2020b): “Technical, Allocative and Overall Efficiency: Estimation and Inference,” *European Journal of Operational Research*, 282(3), 1164–1176.

Simar, L., and V. Zelenyuk (2006): "On Testing Equality of Distributions of Technical Efficiency Scores," *Econometric Reviews*, 25(4), 497–522.

Tulkens, H. (1993): "On FDH Efficiency Analysis: Some Methodological Issues and Applications to Retail Banking, Courts, and Urban Transit," *Journal of Productivity Analysis*, 4(1-2), 183–210.

Wang, T., D. Song, and K. Cullinane (2003): "Container Port Production Efficiency: A Comparative Study of DEA and FDH Approaches," *Journal of the Eastern Asia Society for Transportation Studies*, 5, 698–713.

Wilson, P. (2021): "U.S. Banking in the Post-Crisis Era: New Results from New Methods," in *Advances in Efficiency and Productivity Analysis*, ed. by C. Parmeter, and R. Sickles, pp. 233–264. Springer, Berlin.

Wilson, P. W., and S. Zhao (2023a): "Investigating the Performance of Chinese Banks over 2007–2014," *Annals of Operations Research*, 321(1-2), 663–692.

Wilson, P. W., and S. Zhao (2023b): "Investigating the Performance of Chinese Banks over 20072014," *Annals of Operations Research*, 321(1–2), 663–692.

Wilson, P. W., and S. Zhao (2025): "A Non-Parametric Analysis of World Productivity Growth, 1990–2019," *Annals of Operations Research*, 346(3), 2253–2285.

Appendix A Adapted Li-Test

Simar and Zelenyuk (2006) extend the Li-type test for the equality of distributions to the context of technical efficiency. In this section, we outline how to extend the Simar and Zelenyuk (2006) test from the context of technical efficiency to the settings of cost efficiency and allocative efficiency, focusing on Algorithm II in Simar and Zelenyuk (2006).

Simar and Wilson (2020b) show that under convexity cost efficiency estimated using a traditional convex VRS technology converges at the rate $\kappa_1 = n^{2/(s+2)}$, while under nonconvexity the cost efficiency estimated using the nonconvex VRS technology converges at the rate $\kappa_2 = n^{1/(s+1)}$, where s is the number of outputs. Since the convex and nonconvex VRS cost efficiency estimators have different convergence rates, we follow the logic of Simar and Zelenyuk (2006) and apply different smoothing methods on the cost efficiency estimates before implementing the traditional Li-test. For cost efficiency under a convex VRS technology, we use the following smoothing method:

$$1/CE_i^{C,VRS,*} = \begin{cases} 1/CE_i^{C,VRS} + \varepsilon_1, & \text{if } CE_i^{C,VRS} = 1, \\ \\ 1/CE_i^{C,VRS}, & \text{otherwise} , \end{cases} \tag{A1}$$

where $\varepsilon_1 = \text{Uniform}(0, \min(n^{-\kappa_1}, \epsilon_1 - 1))$ and ϵ_1 is the 5%-quantile of the empirical distribution of $1/CE_i^{C,VRS} > 1$.

For cost efficiency under nonconvex VRS technology, we use the following smoothing method:

$$1/CE_i^{NC,VRS,*} = \begin{cases} 1/CE_i^{NC,VRS} + \varepsilon_2, & \text{if } CE_i^{NC,VRS} = 1, \\ \\ 1/CE_i^{NC,VRS}, & \text{otherwise} , \end{cases} \tag{A2}$$

where $\varepsilon_2 = \text{Uniform}(0, \min(n^{-\kappa_2}, \epsilon_2 - 1))$ and ϵ_2 is the 5%-quantile of the empirical distribution of $1/CE_i^{NC,VRS} > 1$.

Once we have the smoothed estimates for cost efficiency under convex and nonconvex VRS technologies, then we use the Li-test developed by Li, Maasoumi, and Racine (2009) to test the null hypothesis that the distributions of cost efficiency under convex and nonconvex VRS technologies are the same. To be more specific, we use the `npdeneqtest` command in the `np` package of the **R** software. The **R** code to implement the adapted Li-test is available upon request.

For allocative efficiency, Simar and Wilson (2020b) show that under convexity allocative efficiency estimated using VRS converges at the rate $\kappa_3 = n^{2/(m+s+1)}$, while under nonconvexity the allocative efficiency estimated using the traditional VRS converges at the rate $\kappa_4 = n^{1/(m+s)}$, where m is the number of inputs. As the convex and nonconvex VRS estimators for allocative efficiency have different convergence rates, we use different smoothing methods. For allocative efficiency under a convex VRS technology, we use the following smoothing method:

$$1/AE_i^{C,VRS,*} = \begin{cases} 1/AE_i^{C,VRS} + \varepsilon_3, & \text{if } AE_i^{C,VRS} = 1, \\ \\ 1/AE_i^{C,VRS}, & \text{otherwise ,} \end{cases} \tag{A3}$$

where $\varepsilon_3 = \text{Uniform}(0, \min(n^{-\kappa_3}, \epsilon_3 - 1))$ and ϵ_3 is the 5%-quantile of the empirical distribution of $1/AE_i^{C,VRS} > 1$.

For allocative efficiency under a nonconvex VRS technology, we use the following smoothing method:

$$1/AE_i^{NC,VRS,*} = \begin{cases} 1/AE_i^{NC,VRS} + \varepsilon_4, & \text{if } AE_i^{NC,VRS} = 1, \\ \\ 1/AE_i^{NC,VRS}, & \text{otherwise ,} \end{cases} \tag{A4}$$

where $\varepsilon_4 = \text{Uniform}(0, \min(n^{-\kappa_4}, \epsilon_4 - 1))$ and ϵ_4 is the 5%-quantile of the empirical distribution of $1/AE_i^{NC,VRS} > 1$.

Once we have the smoothed estimates for allocative efficiency under convex and nonconvex VRS technologies, we then use the Li-test developed by Li, Maasoumi, and Racine (2009) to test the null hypothesis that the distributions of allocative efficiency under convex and nonconvex VRS technologies are the same. Again we use the `npdeneqtest` command in the `np` package of the **R** software. The **R** code to implement the adapted Li-test is available upon request.