August 2026

WORKING PAPER SERIES

2026-EQM-06

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Global Economies of Scale in Nonconvex Cost Functions: New Proposals and Empirical Illustration*

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December 21, 2025
First revision: August 24, 2026

Abstract

Economies of Scale (ES), defined in terms of minimal Ray Average Cost (RAC) and optimal scale size, is a key concept in economics. We refine global ES by considering two new notions, called Global Sub-Increasing ES (G-SIES) and Global Sub-Decreasing ES (G-SDES), which both depend on the behaviour of the RAC function in a nonconvex setting. These notions provide qualitative information about the optimal scale size and position an observation relative to the true flexible returns to scale technology. After presenting a motivating example and defining G-SIES and G-SDES, we characterize these new notions. Then, we calculate a stability interval (region) for the new ES classification against output changes. We establish several technical theorems and design polynomial-time enumeration algorithms which determine the ES of the units under evaluation and provide stability intervals. These results are illustrated by numerical examples and by two empirical studies on real-world data sets from the electricity generation and distribution sectors.

Keywords: *Data Envelopment Analysis; Free Disposal Hull; Sub-global economies of scale; RAC function; Stability.*

Declarations of interest: none

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*We gratefully acknowledge the most constructive comments of three referees. The usual disclaimer applies.

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1 Introduction

Economies of scale (ES) constitute a fundamental concept in production economics and operations research. Broadly speaking, ES describe the extent to which unit production costs decrease as the scale of an operation increases, thereby allowing the adoption of more efficient and often more capital-intensive production methods (McGee 2014). Such cost reductions may arise from specialization, the spreading of fixed costs, technical efficiencies, organizational improvements, and enhanced market power. Because of their direct impact on productivity and competitiveness, ES play a central role in shaping industrial structure, production planning, and strategic decision making.

The analysis of ES has attracted considerable attention in the economics and efficiency-analysis literature. Early studies investigated scale characteristics from a *local* perspective through the behavior of the ray average cost (RAC) function under proportional output expansions. In particular, Panzar & Willig (1977) characterized local ES using the scale elasticity of the cost function. The integration of these ideas into Data Envelopment Analysis (DEA) subsequently led to a variety of approaches for measuring and characterizing local ES under convex nonparametric production technologies; see, for example, Färe & Grosskopf (1985), Sueyoshi (1999), and the references therein.

Despite the substantial body of literature on ES, three important limitations remain. First, most existing studies focus on *local* rather than *global* scale characteristics. As pointed out by Podinovski (2004*a*), local analyses may provide only partial information regarding the scale behavior of decision-making units (DMUs), potentially leading to incomplete classifications and less informed managerial decisions. Second, the vast majority of available methods rely on the convexity assumption. While convexity is mathematically convenient, it may be inappropriate in many practical situations involving indivisibilities, setup costs, or technological discontinuities. As noted by Shephard (1970, p. 15), convexity is justified only for "time divisibly-operable technologies." When time is imperfectly divisible or switching among activities requires positive setup times, nonconvex production technologies naturally arise. Moreover, in nonconvex settings the cost function may fail to be subdifferentiable, rendering elasticity-based approaches ineffective even for local ES analysis (Cesaroni et al. 2017*b*). Third, existing ES classifications rarely address the issue of stability. Determining stability regions for ES classifications can provide decision makers with valuable information regarding the robustness and reliability of the obtained results.

Motivated by these observations, this paper investigates *global* ES in nonconvex multi-input, multi-output production technologies. To address the above research gaps, we make three main contributions. First, we extend the existing taxonomy of global ES by introducing two new classes based on the sub-global behavior of the RAC function along different output directions. We do it under nonconvex production technologies, and provide an economic interpretation of the new ES categories. Second, for this enriched classification framework, we develop stability intervals (regions) with respect to output levels, thereby quantifying the robustness of ES classifications. Third, we propose enumeration algorithms for determining both the ES classifications and their associated stability regions. Together, these developments provide a more comprehensive and practically relevant framework for analyzing scale characteristics in nonconvex production technologies.

This paper is structured as follows. Section 2 is devoted to the literature review. Section 3 introduces the notation, the nonconvex technologies as well as the corresponding cost functions. Section 4 extends the ES classification for nonconvex cost functions with two new sub-global ES notions. The next Section 5 provides an economic interpretation of the new ES categories and shows how the new ES classification can be determined by simple implicit enumeration tests. Section 6 studies the stability intervals for ES under nonconvex technologies when the output vector changes proportionally. Enumeration polynomial-time algorithms and procedures are provided which are able to calculate stability intervals of the global ES classes. Furthermore, these procedures lead to closed formulas of cost functions under nonconvex technologies. We empirically illustrate our theoretical results on two data sets from electricity generation and distribution sectors in Section 7. The final Section 8 concludes.

2 Literature Review

Economies of scale (ES) have been widely studied across various fields. In banking, Beccalli et al. (2015) documented widespread scale economies in European banks, while Wheelock & Wilson (2018) identified persistent increasing returns to scale in U.S. banking, particularly largest institutions. Beyond finance and economics, ES have been studied in diverse fields. Shriya et al. (2026) examine ES in organic agriculture, Casalicchio et al. (2025) in energy communities, Zhao et al. (2026) in university mergers, and Anderson et al. (2002) in real estate investment trusts using DEA.

The extension of the concept of ES to multi-output production can be traced back to the seminal contributions of Baumol (1977) and Panzar & Willig (1977). The first article introduces the ray average cost (RAC) as a value function of the proportional variation in outputs (see Baumol (1977, 810–811)). The second article provides for a generic production technology a formal definition of the degree of scale economies based on the scale elasticity of the cost function with respect to outputs: thus, performing a local analysis of the scale economies at a given output vector/mix (see Panzar & Willig (1977, Theorem 2: p. 490)).

This local approach is extended by Sueyoshi (1999) to nonparametric production technologies characterized by convexity.[1] In this context, the cost function is not necessarily differentiable at some optimal projection points, and the scale elasticity may assume two distinct values (i.e., the eventual presence of "multiple solutions" in his words).

Following another research direction, Tone & Sahoo (2004) propose a nonparametric approach for measuring scale elasticity in the presence of congestion. Bayón et al. (2012) generalize the firm's profit-maximization problem by incorporating ES via quadratic concave cost functions. Daraio et al. (2015) study ES using a directional distance function, and Carvalho & Marques (2014) analyze economies of scope, and ES using partial–frontier nonparametric methods.

[1] In the operations research literature, these models are often known under the name Data Envelopment Analysis (DEA).

A different approach to ES, which we may term global following Podinovski (2004*a*), is introduced by Färe & Grosskopf (1985) for the case of convex nonparametric production technologies. When cost-scale efficiency, i.e., the ratio between the variable returns to scale (VRS) and the constant returns to scale (CRS) cost function values of an efficient point, is less than one, then the nonincreasing returns to scale (NIRS) cost function value is used to ascertain whether scale inefficiency is due to either decreasing or increasing ES. In the first case the optimal scale size is lower than the outputs under evaluation, while in the second case it is larger than the outputs under evaluation. The approach in question is global in the sense that it only determines the direction to the optimal scale size, without supplying quantitative information on the local intensity of scale economies at a given output mix.

Both of these described approaches encounter problems when applied to a basic nonconvex production technology, known as the Free Disposal Hull (FDH) developed by Deprins et al. (1984), or to the extensions of this FDH technology to the case of CRS, NIRS and nondecreasing returns to scale (NDRS) as developed in Kerstens & Vanden Eeckaut (1999).

In particular, as a consequence of the lack of sub-differentiability of the cost function in the nonconvex case, the scale elasticity approach fails to deliver sensible economic information as far as local scale economies are concerned.[2] This outcome is clearly documented for the returns to scale (RTS) analysis based on ray average productivity by Cesaroni et al. (2017*b*, Section 3.2)): obviously, it also holds for the ES analysis. Concerning the global approach, this method of Färe & Grosskopf (1985) fails to detect the case of sub-constant ES, due to the lack of consideration of the cost function under NDRS (see Podinovski (2004*a*) or Cesaroni et al. (2017*a*, Section 2.2)). These modifications are required by the nonconvexity of the cost function in outputs occurring in a nonconvex technology (see Kerstens & Van de Woestyne (2021)).

The greater suitability of the global approach for nonconvex technologies and the possibility of simplifying computations by considering exclusively the RAC function in the VRS technology (see Chavas & Cox (1999), Cesaroni & Giovannola (2015)) has led to the elaboration of the first method of classification of ES discussed in Cesaroni et al. (2017*a*). Kerstens & Van de Woestyne (2021) refine this method by making explicit the relationship between ES and the values of the cost function under different returns to scale assumptions (CRS, VRS, NIRS and NDRS). Moreover, these authors provide a comparison of empirical estimates of global ES obtained under the convex and the nonconvex assumption and report conflicting information. This suggests, as far as decision support is concerned, an empirically significant role of the nonconvexity of the cost function in the outputs, as documented in Kerstens & Van de Woestyne (2021).

Despite the considerable literature on ES, most studies are either context-specific or restricted to local or convex frameworks. Moreover, global scale properties and their stability have received relatively little attention. Consequently, existing approaches may provide only partial classifications of scale behavior. Podinovski & Førsund (2022) provide an up to date survey of the notions of

[2]By contrast, sub-differentiability of the cost function holds for convex nonparametric production technologies and makes available an unambiguous classification of local ES given in Sueyoshi (1999).

scale elasticity and returns to scale for convex and nonconvex technologies. This motivates the development of a more general framework for analyzing *global* ES in *nonconvex* technologies while accounting for the *stability* of the resulting classifications. Specifically, this paper makes three main contributions. First, we introduce two new cases in the global ES classification by examining the sub-global behavior of the RAC in two directions. Second, we determine the corresponding output stability intervals (regions). Third, we develop enumeration algorithms for both the classification and its stability intervals (regions).

3 Notation, Axioms, Technologies and Cost Functions

In this paper, the n-dimensional Euclidean space is denoted by $\mathbb{R}^n$. Given two vectors $x, y \in \mathbb{R}^n$, the vector inequality $x \leq y$ means $x_j \leq y_j$ for each $j = 1, 2, \ldots, n$; and similarly for $<, \geq$, and $>$. The zero vector (matrix) is denoted by 0 without mentioning its dimension; it is understood from the context. The set of nonnegative vectors is $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : \; x \geq 0\}$. Furthermore, $\mathbb{R}^n_{++} = \{x \in \mathbb{R}^n : \; x > 0\}$. The vector with all components equal to one is denoted by e.

Suppose that we have a set of $n \geq 2$ DMUs consisting of DMU_j; $j \in J = \{1, \ldots, n\}$, with input-output vectors (x_j, y_j); $j \in J$. Each DMU_j consumes m positive inputs $x_{1j}, x_{2j}, \ldots, x_{mj} > 0$ to produce s positive outputs $y_{1j}, y_{2j}, \ldots, y_{sj} > 0$.[3] Define $x_j := (x_{1j}, x_{2j}, \ldots, x_{mj})^T$ and $y_j := (y_{1j}, y_{2j}, \ldots, y_{sj})^T$ as input and output vectors of DMU_j, respectively. The superscript "T" stands for transpose.

The production possibility set or technology T is defined as $T = \{(x, y) \in \mathbb{R}^m_+ \times \mathbb{R}^s_+ : \; x$ can produce $y\}$. The technology investigated in the current paper satisfies some combination of the following convenient assumptions:

(T1) $(0, 0) \in T$. Furthermore, $(0, y) \in T \Longrightarrow y = 0$.

(T2) T is closed.

(T3) If $(x, y) \in T$ and $(x', y') \in \mathbb{R}^m_+ \times \mathbb{R}^s_+$, then $(x', -y') \geq (x, -y) \Longrightarrow (x', y') \in T$.

(T4) $(x, y) \in T \Longrightarrow \delta(x, y) \in T$ for $\delta \in \Delta$, where:

- (i) $\Delta = \Delta^{CRS} = \{\delta : \; \delta \geq 0\}$;
- (ii) $\Delta = \Delta^{NDRS} = \{\delta : \; \delta \geq 1\}$;
- (iii) $\Delta = \Delta^{NIRS} = \{\delta : \; 0 \leq \delta \leq 1\}$;
- (iv) $\Delta = \Delta^{VRS} = \{\delta : \; \delta = 1\}$,

[3] The proposed approach can in principle accommodate zero-valued inputs and outputs: to avoid additional complexity in notation and proofs, we ignore this possibility in this work. In FDH models and enumeration algorithms, zero data may produce undefined expressions of the form (0/0), but these can be readily detected and handled through simple enumeration tests; see Soleimani-damaneh et al. (2006, p. 1057). In practice, however, zero data are uncommon because comparable units typically consume positive amounts of all inputs and produce positive amounts of all outputs. This is also true for the two data sets used in our empirical illustration.

where CRS, NDRS, NIRS, and VRS stand for constant, non-decreasing, non-increasing, and variable Returns to Scale (RTS), respectively. The above-mentioned axioms can be called as (T1) inaction is possible, and no free lunch, (T2) closedness, (T3) free disposal of inputs and outputs, and (T4) returns to scale assumption. Some of the postulates can be ignored in some practical situations (see, e.g., Kerstens & Van de Woestyne (2021, p. 3)).[4]

There is another traditional technology axiom, called *convexity*, as follows:

(T5) T is a convex subset of $\mathbb{R}^m_+ \times \mathbb{R}^m_s$.

This assumption is almost universally applied. But, it is not without controversy: Shephard (1970, p. 15) states that convexity is only valid for "time divisibly-operable technologies". Thus, if time is imperfectly divisible and switching between activities entails a positive setup time, then convexity does not hold. In brief, convexity is an empirical assumption that ideally requires testing. Thus, in the current paper, we ignore this axiom and work mainly with nonconvex technologies to derive our theoretical results.

A well-known nonconvex technology is the Free Disposal Hull (FDH), first developed by Deprins et al. (1984). Indeed, an FDH technology is the smallest subset of $\mathbb{R}^m_+ \times \mathbb{R}^m_s$ satisfying (T1)-(T4). Several representations of FDH technologies can be found in the literature. We utilize the following representation, under different RTS assumptions, first investigated by Kerstens & Vanden Eeckaut (1999):

$$T^{FDH_\Delta} = \{(x,y) : \sum_{j\in J} \mu_j x_j \leq x,\ \sum_{j\in J} \mu_j y_j \geq y \geq 0,\ \mu = \delta\gamma,\ \sum_{j\in J} \gamma_j = 1,\ \gamma \in \{0,1\}^n,\ \delta \in \Delta\}.$$

Since γ_j variables are binary, due to the constraints $\sum_{j\in J} \gamma_j = 1$ and $\mu_j = \delta\gamma_j$, all but one of the μ_j variables will be zero. This implies that efficiency evaluation by FDH models is affected from only actually observed performances. This is one of the most important characteristics of FDH models. The RTS of the production technology is controlled by δ. Hereafter, for simplicity, we use the notations "V", "C", "NI", and "ND", instead of "VRS", "CRS", "$NIRS$", and "$NDRS$", respectively.

Consider DMU$_o = (x_o, y_o)$ with $o \in J$ as the unit under assessment. The input-oriented FDH radial efficiency measures of DMU_o are obtained by solving the following mixed-integer nonlinear programming problem (Kerstens & Vanden Eeckaut (1999)):

$$\begin{aligned} \phi_o^\Delta = \quad \min \quad \phi \quad s.t. \quad & \sum_{j\in J} \mu_j x_j \leq \phi x_o, \quad \sum_{j\in J} \mu_j y_j \geq y_o, \\ & \mu_j = \delta\gamma_j, \quad \gamma_j \in \{0,1\};\ \forall j \in J, \quad \sum_{j\in J} \gamma_j = 1,\ \delta \in \Delta,\ \phi \geq 0, \end{aligned} \tag{1}$$

[4] As requested by a referee, it is in principle possible to weaken the free disposal assumption (T3) to model, e.g., congestion. In a cost function context this leads, among others, to potentially negative inputs prices (see Shephard (1974)). But, this has so far remained a marginal topic in cost function analysis. Therefore, we ignore these developments in our contribution.

where $\Delta \in \{\Delta^V, \Delta^C, \Delta^{NI}, \Delta^{ND}\}$. We use the notation (1-V) for referring to model (1) under a variable RTS assumption, and analogously for other RTS types and related models. Furthermore, we utilize the notations ϕ_o^V, ϕ_o^C, ϕ_o^{NI}, and ϕ_o^{ND} instead of $\phi_o^{\Delta^V}$, $\phi_o^{\Delta^C}$, $\phi_o^{\Delta^{NI}}$, and $\phi_o^{\Delta^{ND}}$, respectively.

Let $w \in \mathbb{R}^m_{++}$ be a given input price row vector. Furthermore, let $y \in \mathbb{R}^s_{++}$ be a given feasible output vector. The vector $y \in \mathbb{R}^s_{++}$ is called feasible in technology T, if $(x, y) \in T$ for some $x \in \mathbb{R}^m_+$. The cost function, which addresses the FDH cost efficiency at the output level y in technology T^Δ, is defined as $C(y, w|\Delta) = \min\{wx : (x, y) \in T^\Delta\}$. Here, T^Δ stands for technology with $\delta \in \Delta$. Considering the FDH technologies, $C(y, w|\Delta)$ is the optimal value of the following mixed-integer nonlinear programming problem:

$$C(y,w|\Delta) = \min \quad wx \quad s.t. \quad \sum_{j\in J} \mu_j x_j \le x, \quad \sum_{j\in J} \mu_j y_j \ge y, \qquad \mu_j = \delta\gamma_j, \quad \gamma_j \in \{0,1\};\ \forall j \in J, \quad \sum_{j\in J} \gamma_j = 1,\ \delta \in \Delta. \tag{2}$$

There are interesting connections between two models (1) and (2) (see Hackman (2008)).

By setting $X := [x_1, \dots, x_n]$ and $Y := [y_1, \dots, y_n]$, these two matrices are $m \times n$ and $s \times n$, respectively, and model (2) can be written as

$$C(y,w|\Delta) = \min\Big\{wx :\ X\mu \le x,\ Y\mu \ge y,\ \mu = \delta\gamma,\ e\gamma = 1,\ \gamma \in \{0,1\}^n,\ \delta \in \Delta\Big\}.$$

Recall that e is the vector with all components equal to one. Similarly, model (1) can be written in a more concise way.

4 Economies of Scale: A New Classification

In this section, we investigate the ES defined in terms of the cost function in nonconvex technologies (see Kerstens & Van de Woestyne (2021)), and we introduce in the existing classification two new ES notions that can be labeled as sub-global ES notions. To this end, we need the minimal RAC concept developed in Chavas & Cox (1999) and Cesaroni & Giovannola (2015).

Definition 1. For a given feasible output mix $y \in \mathbb{R}^s_{++}$ and an input price vector $w \in \mathbb{R}^m_{++}$:

(i) the Ray Average Cost (RAC) function with respect to technology T^Δ, is a function $RAC : (0, +\infty) \longrightarrow (0, +\infty)$ defined as:

$$RAC(y,w|\Delta)(\lambda) = \frac{C(\lambda y, w|\Delta)}{\lambda}.$$

(ii) the Minimum Ray Average Cost ($MinRAC$) with respect to technology T^Δ is defined as:

$$MinRAC(y,w|\Delta) = \min_\lambda \left\{\frac{C(\lambda y, w|\Delta)}{\lambda} :\ \lambda > 0\right\} = \min\Big\{RAC(y,w|\Delta)(\lambda) :\ \lambda > 0\Big\}.$$

Notice that the domain of the RAC function is $\left\{\lambda \in (0, +\infty) : \exists x \in \mathbb{R}^m_+ \text{ s.t. } (x, \lambda y) \in T^\Delta\right\}$. The optimal value of λ realizing MinRAC (i.e., λ^*) determines the optimal scale size ($\lambda^* y$). The following Lemma 1 provides some connections between the RAC and the cost function.

Lemma 1. Chavas & Cox (1999), Kerstens & Van de Woestyne (2021)

(a) $MinRAC(y, w|C) = MinRAC(y, w|V) = C(y, w|C)$.

(b) $MinRAC(y, w|\Delta) \leq C(y, w|\Delta)$.

There are several papers in the literature discussing the local and global RTS in convex and nonconvex technologies; see, e.g., Banker & Thrall (1992), Cooper et al. (2007), Kerstens & Vanden Eeckaut (1999), Krivonozhko & Lychev (2019), Podinovski (2004*b*), Soleimani-damaneh et al. (2006) among others. To the best of our knowledge, the first proposal of a method for global ES based on the values of λ is given by Cesaroni & Giovannola (2015). A complete method for the current ES classification is first developed in Cesaroni et al. (2017*a*). The most recent work providing a more general characterization of ES is found in Kerstens & Van de Woestyne (2021). These two latter contributions provide the following definition, whereby CES, IES, DES, and SCES stand for constant, increasing, decreasing, sub-constant economies of scale.

Definition 2. Cesaroni et al. (2017*a*), Kerstens & Van de Woestyne (2021) Let the feasible output mix $y \in \mathbb{R}^s_{++}$ and the price vector $w \in \mathbb{R}^m_{++}$ be given. Then, we can state:

(i) CES prevails if $MinRAC(y, w|V)$ is attained for at least the optimal scale size 1;

(ii) IES prevails if $MinRAC(y, w|V)$ is attained for one or more optimal scale sizes, all of which are strictly greater than 1;

(iii) DES prevails if $MinRAC(y, w|V)$ is attained for one or more optimal scale sizes, all of which are strictly smaller than 1;

(iv) SCES prevails if $MinRAC(y, w|V)$ is attained for multiple optimal scale sizes, some of which are strictly larger than 1 and others are strictly smaller than 1.

Now, we continue with a motivating example toward introducing two new sub-global ES concepts according to the different behaviour of the RAC function in the two branches $\lambda > 1$ and $\lambda < 1$.

Example 1. Consider a set of six DMUs consisting of $\text{DMU}_A, \ldots, \text{DMU}_F$, in which each DMU utilizes a single input to produce a single output. The data of these DMUs are listed in Table 1. The frontier of the technology with various RTS assumptions can be seen in Figure 1. The frontier in VRS case has been depicted by the dark line segments. The frontier under CRS assumption is the halfline starting from the origin passing through the points D and F. The frontier under NDRS assumption is the union of the line segments $\overline{HA} \cup \overline{AP} \cup \overline{PD}$ and the halfline starting from D

DMUs	A	B	C	D	E	F
x	1	3	6	7	11	14
y	1	3	5	10	12	20

Table 1: Data in Example 1.

passing through F. Moreover, the frontier under NIRS assumption is the union of the line segment $\overline{OF}$ and the halfline $\{F + \lambda(1,0)^T : \lambda \geq 0\}$ (recall that superscript "T" stands for transpose).

Assume that $w = 1$. We focus on the increasing ES area. If we consider $y = 1$ (observed output of DMU$_A$), then from Figure 1 it is not difficult to see that the RAC function will be as follows. The domain of this function is $(0, 20]$.

$$RAC(y = 1, w = 1|V)(\lambda) = \begin{cases} \frac{1}{\lambda}, & 0 < \lambda \leq 1, \\ \frac{3}{\lambda}, & 1 < \lambda \leq 3, \\ \frac{6}{\lambda}, & 3 < \lambda \leq 5, \\ \frac{7}{\lambda}, & 5 < \lambda \leq 10, \\ \frac{11}{\lambda}, & 10 < \lambda \leq 12, \\ \frac{14}{\lambda}, & 12 < \lambda \leq 20 \end{cases}$$

The graph of this function is depicted in Figure 2. Figures 3 and 4 exhibit the RAC function with $y = 3$ (observed output of DMU$_B$) and $y = 5$ (observed output of DMU$_C$), respectively. The domains of the last two functions are $(0, \frac{20}{3}]$ and $(0, 4]$, respectively.

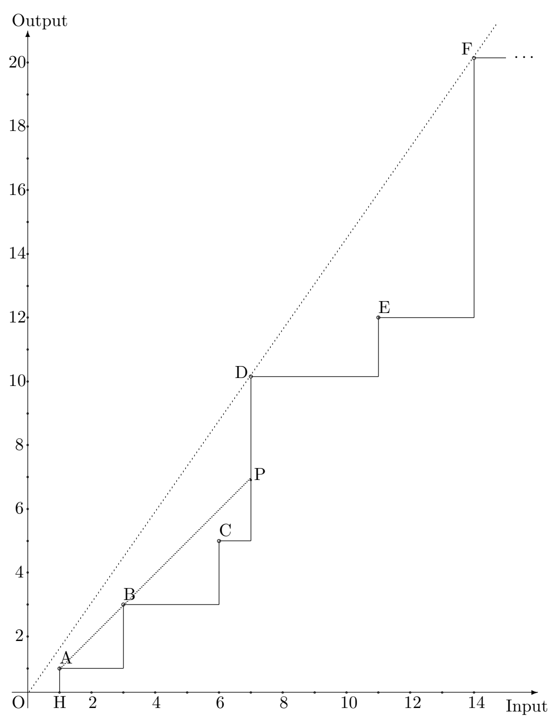

Figure 1: FDH Technologies for Example 1

From Figures 2-4, it is observed that for all three output levels $y = 1,\ 3,\ 5$, minimal ray average cost, $MinRAC(y, w|V) = \min\left\{RAC(y, w|V)(\lambda) :\ \lambda > 0\right\}$, is attained for optimal scale sizes, all of which are strictly greater than 1. For example, for $y = 1$, $MinRAC(y, w|V) = \frac{7}{10}$ is attained for $\lambda = 10$ and $\lambda = 20$. So, according to Definition 2, for all three output levels $y = 1,\ 3,\ 5$ (for three units A, B, C) IES prevails. But there is an important difference between the behaviour of the RAC function corresponding to these three output levels at its minimizers.

Consider $y = 1$ with IES status (in the sense of Definition 2), and its corresponding RAC function depicted in Figure 2 (denoted RAC_A). The minimum of RAC_A over the left side of $\lambda = 1$ (including 1) is greater than the minimum of RAC_A over the right side of $\lambda = 1$ (including 1); and the former happens at $\lambda = 1$.

Now, consider $y = 3$ with IES status again. Its corresponding RAC function depicted in Figure 3 (denoted RAC_B) behaves similar to RAC_A.

Now, consider $y = 5$ (observed at DMU_C) with IES status again, and its corresponding RAC function (denoted RAC_C) depicted in Figure 4. It is seen that the minimum of RAC_C over the left side of $\lambda = 1$ (including 1) is greater than the minimum of RAC_C over the right side of $\lambda = 1$ (including 1); while none happens at $\lambda = 1$.

This comparison highlights the difference of the behavior of RAC (from minimum standpoint) in different output levels with the same IES status. Taking this difference into account, we define a new concept called Global Sub-Increasing ES (G-SIES): see Definition 3. In fact, when RAC function behaviour is similar to RAC_A and RAC_B, we say that G-IES happens. When RAC function behaviour is similar to RAC_C, we say that G-SIES happens.

An analogous analysis and comparison can be done for decreasing ES as well. Such an analysis and comparison lead to defining a new concept called Global Sub-Decreasing ES (G-SDES): see Definition 3. Based on our new classification, for the output level with G-DES, the minimum of RAC function over the left side of $\lambda = 1$ (including 1) is less than its minimum over the right side of $\lambda = 1$ (including 1); and the latter happens at $\lambda = 1$. For the output levels with G-SDES, the minimum of the RAC function over the left side of $\lambda = 1$ (including 1) is less than its minimum over the right side of $\lambda = 1$ (including 1); while none happens at $\lambda = 1$.

A similar discussion can be provided for CES and SCES. We skip it here: see parts (i) and (ii) of Definition 3.

Now, we are ready to present our new Definition 3.

Definition 3. Let the feasible output level $y \in \mathbb{R}^s_{++}$ and the input price vector $w \in \mathbb{R}^m_{++}$ be given. Then, we say:

(i) G-CES prevails if the minimum of RAC over $(0, 1]$ is equal to the minimum of RAC over $[1, \infty)$; and both happen at $\lambda = 1$.

(ii) G-SCES prevails if the minimum of RAC over $(0, 1]$ is equal to the minimum of RAC over $[1, \infty)$; while none happens at $\lambda = 1$.

(iii) G-IES prevails if the minimum of RAC over $(0, 1]$ is greater than the minimum of RAC over $[1, \infty)$; and the former happens at $\lambda = 1$.

(iv) G-SIES prevails if the minimum of RAC over $(0, 1]$ is greater than the minimum of RAC over $[1, \infty)$; while none happens at $\lambda = 1$.

(v) G-DES prevails if the minimum of RAC over $(0, 1]$ is less than the minimum of RAC over $[1, \infty)$; and the latter happens at $\lambda = 1$.

(vi) G-SDES prevails if the minimum of RAC over $(0, 1]$ is less than the minimum of RAC over $[1, \infty)$; while none happens at $\lambda = 1$.

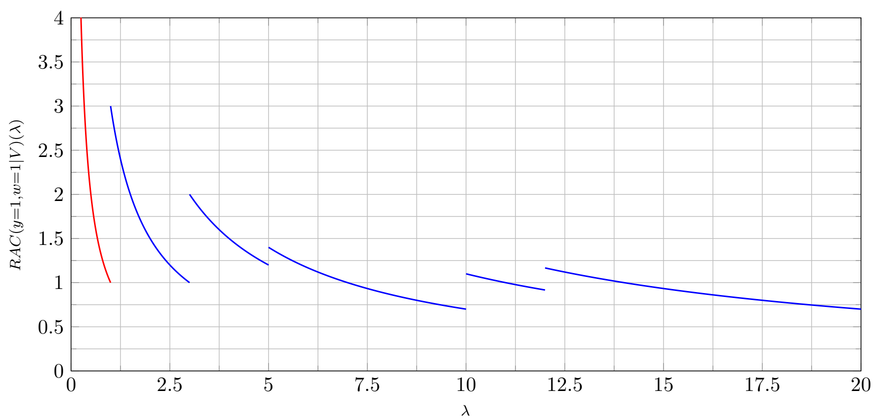

Figure 2: RAC_A Function for $y = 1$.

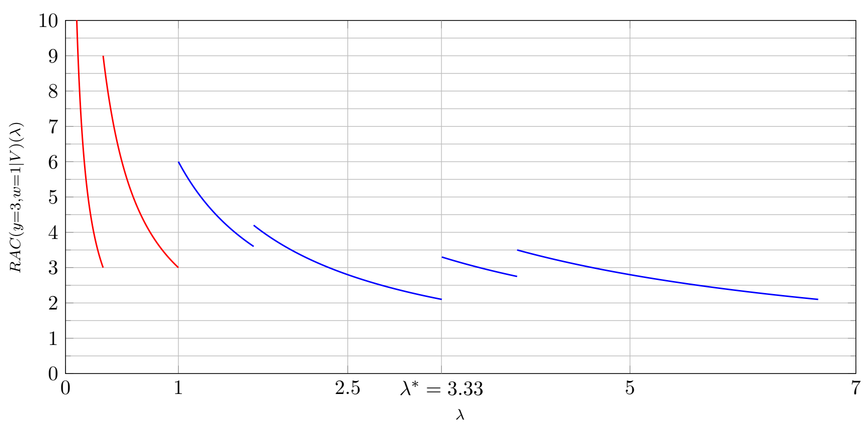

Figure 3: RAC_B Function for $y = 3$.

Note that the two new sub-global notions, G-SIES and G-SDES, are obtained as refinements of the standard G-IES and G-DES classifications. In fact, the new G-SIES and G-SDES cases are classified as G-IES and G-DES, respectively, in the standard ES classification, while they are now given separate evidence in the newly proposed classification in the nonconvex case. Note that Cesaroni et al. (2026) show by analysing RAC functions defined on convex technologies that the above cases of sub-increasing, sub-constant and sub-decreasing SE cannot occur, because RAC is either monotonically increasing or monotonically decreasing at any point which is not an optimal scale size.

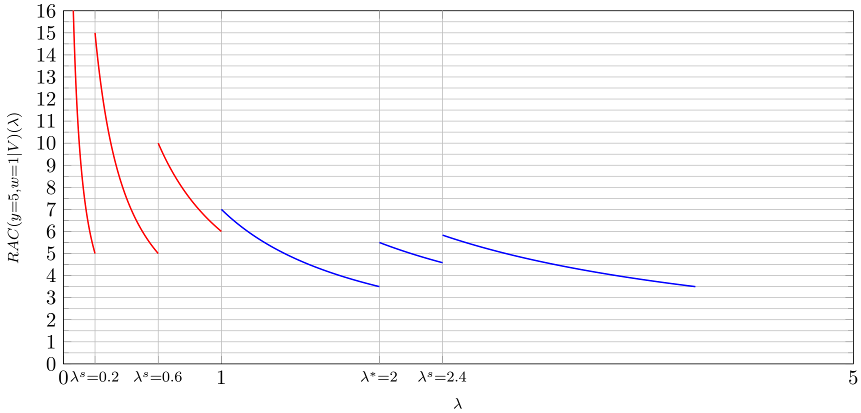

Figure 4: RAC_C Function for $y = 5$.

5 Determination of the New ES Classification

The first aim of this section is to provide an economic interpretation of the new ES categories. In the increasing economies of scale (IES) region, the minimum RAC over $(0, 1]$ is greater than the minimum RAC over $[1, \infty)$. This means that the lowest average cost attainable by reducing the scale of activity is higher than the lowest average cost attainable by expanding the scale. Therefore, increasing the scale beyond the current output level allows the firm to achieve a lower average cost than any contraction of the production scale. Economically, the current output level y is below the cost-minimizing scale, and the firm can reduce its average cost by operating at a larger scale.

However, the IES property alone does not reveal whether some (possibly suboptimal) contraction of the scale can also improve the current average cost. In other words, although the optimal average cost is achieved on the expansion side, it remains an open question whether there exists a contracted scale $\lambda < 1$ such that $RAC(y, w|V)(\lambda) < RAC(y, w|V)(1)$. To address this issue, we introduce a new subclass of IES, called *global sub-increasing economies of scale (G-SIES)*. This class characterizes situations in which the current scale is dominated by both a smaller and a larger scale. Specifically, y belongs to the G-SIES class if the minimum of RAC over $(0, 1]$ is greater than the minimum of RAC over $[1, \infty)$, while neither minimum is attained at $\lambda = 1$. Consequently, there exist $\lambda_1 < 1$ and $\lambda_2 > 1$ such that $RAC(y, w|V)(\lambda_1) < RAC(y, w|V)(1)$ and $RAC(y, w|V)(\lambda_2) < RAC(y, w|V)(1)$, with $RAC(y, w|V)(\lambda_2) < RAC(y, w|V)(\lambda_1)$. Thus, in the G-SIES case, a contraction of the scale already improves average cost, while an expansion provides an even greater improvement. The current scale is therefore not a local minimum of the RAC function; it is dominated by both smaller and larger scales, although the larger scale is ultimately the most cost-efficient.

Conversely, if y belongs to the G-IES class but not to the G-SIES class, then the minimum of

RAC over $(0,1]$ is greater than the minimum of RAC over $[1,\infty)$, and the first minimum is attained at $\lambda = 1$. Hence, $RAC(y,w|V)(1) \leq RAC(y,w|V)(\lambda)$ for each $0 < \lambda \leq 1$. In this case, reducing the scale cannot improve average cost; the current scale is already optimal among all contraction possibilities. Nevertheless, expansion can further reduce average cost. Economically, the firm is located at the boundary between contraction and expansion: there is no benefit from becoming smaller, but there is a benefit from becoming larger.

As an example of G-SIES, in Example 1, for $y = 5$, whose corresponding RAC function is depicted in Figure 4, both scales $\lambda^* = 2$ and $\lambda^s = 0.6$ lead to an improvement in the average cost, while the former belongs to the expansion side and the latter to the contraction side. The former is optimal, whereas the latter is not optimal, but it still provides an improvement compared with the current scale (it is called suboptimal or relatively optimal). For $y = 3$, whose corresponding RAC function is depicted in Figure 3, the situation is different. All scales that reduce the average cost are located on the expansion side, and contraction does not lead to any improvement in the average cost.

Indeed, the G-SIES class identifies situations in which a richer set of managerial choices is available regarding the scale of activity of the firm, where both optimal and suboptimal scale adjustments exist in the expansion and contraction directions. Mutatis mutandis, a similar discussion can be provided for the DES area as well.

This interpretation is especially relevant from the short/medium run point of view. In fact, if it can be reasonable to assume that in the long run the adjustment costs to the optimal level of activity determined by the absolute minimum of RAC are negligible (examples are labor force turnover and plant substitution), then the same is not true within a shorter time horizon. In this setting, not only adjustment costs are a function of the magnitude of the output scale variation $(\lambda - 1)$, but these cost may also differ in the increasing vs the decreasing direction, due to the specific adjustment needed for each different input, as the change in input proportions is a significant component of RAC minimization (see Cesaroni & Giovannola (2015, section 3.2).

Besides these consequences for production decisions and cost minimization behaviour, the new global classification may have implications also for the industry structure issue. In a setting with identical individual FDH technologies, Cesaroni (2025, p. 1304) shows that a centralized resource allocation (CRA) minimizing the total cost implements Baumol et al. (1982) concept of "efficient number of firms", that we denote as N. This allocation minimizes the RAC of industry's total output and clearly denotes a long-term notion involving structural change within a sector or industry. Therefore, the kind of cost scale inefficiency (i.e., increasing or decreasing) exhibited by each of the n component DMUs contributes to the determination of the long-term divergence between n and N. Suppose that there is a prevalence of DMUs with increasing scale economies (ISE), then in a CRA we expect to obtain an efficient number of firms $N < n$. However, if these ISE DMUs are of the G-SIES type, due to short/medium term adjustment costs they could find it more appropriate to select a relative minimum in the output decreasing direction, thus significantly reducing the difference between n and $\tilde{N}$, where the latter denotes the suboptimal number of firms determined in the short/medium term, leading, e.g., to $n < \tilde{N} < N$. Other hypothetical cases can be illustrated (e.g., $\tilde{N} < n < N$), but the conclusion is similar: the presence of a large number of DMUs of the G-SIES and G-SDES types may substantially affect the relationship between the actual, the suboptimal and the efficient number of firms in an industry.

The second aim of this section is to show that the new ES, in the sense of Definition 3, can be determined by some simple enumeration tests (see Briec & Kerstens (2006), Kerstens & Van de Woestyne (2018), Krivonozhko & Lychev (2017), Tulkens (1993), Soleimani-damaneh et al. (2006) for more details about enumeration algorithms in FDH models). The following theorem helps us in this regard.

Theorem 1. Let the feasible output level $y \in \mathbb{R}^s_{++}$ and the input price vector $w \in \mathbb{R}^m_{++}$ be given.

(i) G-CES prevails if and only if $C(y, w|NI) = C(y, w|ND) = C(y, w|V)$.

(ii) G-SCES prevails if and only if $C(y, w|NI) = C(y, w|ND) < C(y, w|V)$.

(iii) G-IES prevails if and only if $C(y, w|NI) < C(y, w|ND) = C(y, w|V)$.

(iv) G-SIES prevails if and only if $C(y, w|NI) < C(y, w|ND) < C(y, w|V)$.

(v) G-DES prevails if and only if $C(y, w|ND) < C(y, w|NI) = C(y, w|V)$.

(vi) G-SDES prevails if and only if $C(y, w|ND) < C(y, w|NI) < C(y, w|V)$.

The proof of this theorem and all other theorems is found in Appendix A.

To obtain $C(y, w|NI)$, $C(y, w|ND)$, and $C(y, w|V)$, enumeration algorithms can be used, similar to those developed and applied in Kerstens & Van de Woestyne (2014), Mostafaee & Soleimani-Damaneh (2020), Soleimani-damaneh et al. (2006) which work using vector dominance reasoning. Using some results analogous to those provided in aforementioned publications, we have:

$$C(y, w|ND) = \min_{j \in J:\, \mu_j^* \geq 1} \mu_j^* w x_j, \quad \text{and} \quad C(y, w|NI) = \min_{j \in J:\, \mu_j^* \leq 1} \mu_j^* w x_j,$$

where: $\mu_j^* = \max_{1 \leq r \leq s} \dfrac{y_r}{y_{rj}}$. Furthermore, we have:

$$C(y, w|V) = \min_{j \in J:\, \mu_j^* \leq 1} w x_j.$$

Using the above formulas and Theorem 1, determination of the ES at a given feasible output level $y \in \mathbb{R}^s_{++}$, with a given input cost vector $w \in \mathbb{R}^m_{++}$, is easy. This calculation is polynomial-time. The procedure derived from Theorem 1 is a goodness-of-fit method in the sense of Kerstens & Vanden Eeckaut (1999). See also the discussion of Algorithm 1 below.

Computational complexity. The initialization step of Algorithm 1 requires constant time, i.e., $O(1)$. The main loop is executed n times. For each observation j, the scaling factor μ_j is computed in $O(s)$ time. Furthermore, the cost term wx_j requires $O(m)$ arithmetic operations. The subsequent updates of $C(y, w|V)$, $C(y, w|NI)$, and $C(y, w|ND)$ involve only a constant number of comparisons and assignments and therefore require $O(1)$ time. Hence, each iteration of the loop has complexity $O(m + s)$. Since the loop is repeated for all n observations, the overall computational complexity of the algorithm is $O\big(n(m + s)\big)$. The final classification step requires only a constant number of comparisons and does not affect the asymptotic complexity. Since in practice, $n \geq 3(m + s)$, it follows that $m + s \leq n/3$, and thus the complexity is bounded above by $O(n^2)$. However, the tighter expression $O(n(m + s))$ is retained to reflect the dependence on the numbers of inputs and outputs. Throughout the complexity analysis, constant factors are ignored in Big-O notation. In summary, Algorithm 1 runs in polynomial time with respect to the number of observations.

We continue the current section by implementing Theorem 1 on the data of Example 1. Here, we assume $w = 1$.

The results listed in Table 2 verify the discussions presented in the current and preceding sections. From Table 2, it is observed that G-IES prevails at output levels $y = 1$ (corresponding to DMU A) and $y = 3$ (corresponding to DMU B), while G-SIES prevails at output level $y = 5$ (corresponding to DMU C). Furthermore, the output levels $y = 10,\ 20$ (corresponding to DMUs $D,\ F$, respectively) have G-CES status. Moreover, G-SCES prevails at output level $y = 12$ (corresponding to DMU E). In this example, we do not have G-DES/G-SDES observations, though constructing examples containing these cases is easy.

Proposition 1 can be directly derived from the proof of Theorem 1. Hereafter, as used in the proof of Theorem 1,

$$RAC_- = \min\Big\{RAC(y, w|V)(\lambda) :\ \lambda \in (0, 1]\Big\},\ RAC_+ = \min\Big\{RAC(y, w|V)(\lambda) :\ \lambda \geq 1\Big\}.$$

Proposition 1. Let the feasible output $y \in \mathbb{R}^s_{++}$ and the input price vector $w \in \mathbb{R}^m_{++}$ be given.

(i) δ^* is a part of an optimal solution of (2-ND) if and only if $\lambda^* = \frac{1}{\delta^*}$ is an optimal solution of the problem corresponding to RAC_-.

(ii) δ° is a part of an optimal solution of (2-NI) if and only if $\lambda^\circ = \frac{1}{\delta^\circ}$ is an optimal solution of the problem corresponding to RAC_+.

Given a feasible output level $y \in \mathbb{R}^s_{++}$ and the input price vector $w \in \mathbb{R}^m_{++}$, to determine the ES class it is sufficient to obtain $C(y, w|NI)$, $C(y, w|ND)$, and $C(y, w|V)$, and then apply Theorem 1. Taking this point into account and applying an enumeration approach, we design Algorithm 1 for ES determination, where r is the index of outputs.

6 Stability of GES for Output Changes: Theory and Algorithms

This section investigates the stability intervals (regions) for GES under nonconvex technologies when the output vector changes proportionally. First, let us define the stability interval for preserving the GES status at a specified output mix.

$y = 1:$

j	μ_j^*	$\mu_j^* wx_j$	wx_j
A	1	$1 = C(y, w\|ND)$	$1 = C(y, w\|V)$
B	$\frac{1}{3}$	1	3
C	$\frac{1}{5}$	$\frac{6}{5}$	6
D	$\frac{1}{10}$	$\frac{7}{10}$	7
E	$\frac{1}{12}$	$\frac{11}{12}$	11
F	$\frac{1}{20}$	$\frac{14}{20} = C(y, w\|NI)$	14
			G-IES

$y = 3:$

j	μ_j^*	$\mu_j^* wx_j$	wx_j
A	3	$3 = C(y, w\|ND)$	1
B	1	3	$3 = C(y, w\|V)$
C	$\frac{3}{5}$	$\frac{18}{5}$	6
D	$\frac{3}{10}$	$\frac{21}{10}$	7
E	$\frac{3}{12}$	$\frac{33}{12}$	11
F	$\frac{3}{20}$	$\frac{21}{10} = C(y, w\|NI)$	14
			G-IES

$y = 5:$

j	μ_j^*	$\mu_j^* wx_j$	wx_j
A	5	$5 = C(y, w\|ND)$	1
B	$\frac{5}{3}$	5	3
C	1	6	$6 = C(y, w\|V)$
D	$\frac{5}{10}$	$\frac{35}{10}$	7
E	$\frac{5}{12}$	$\frac{55}{12}$	11
F	$\frac{5}{20}$	$\frac{7}{2} = C(y, w\|NI)$	14
			G-SIES

$y = 10:$

j	μ_j^*	$\mu_j^* wx_j$	wx_j
A	10	10	1
B	$\frac{10}{3}$	10	3
C	2	12	6
D	1	$7 = C(y, w\|ND)$	$7 = C(y, w\|V)$
E	$\frac{10}{12}$	$\frac{55}{6}$	11
F	$\frac{1}{2}$	$7 = C(y, w\|NI)$	14
			G-CES

$y = 12:$

j	μ_j^*	$\mu_j^* wx_j$	wx_j
A	12	12	1
B	4	12	3
C	$\frac{12}{5}$	$\frac{72}{5}$	6
D	$\frac{12}{10}$	$\frac{84}{10} = C(y, w\|ND)$	7
E	1	11	$11 = C(y, w\|V)$
F	$\frac{12}{20}$	$\frac{84}{10} = C(y, w\|NI)$	14
			G-SCES

$y = 20:$

j	μ_j^*	$\mu_j^* wx_j$	wx_j
A	20	20	1
B	$\frac{20}{3}$	20	3
C	4	24	6
D	2	$14 = C(y, w\|ND)$	7
E	$\frac{20}{12}$	$\frac{55}{3}$	11
F	1	$14 = C(y, w\|NI)$	$14 = C(y, w\|V)$
			G-CES

Table 2: GRS determination in Example 1.

Definition 4. Let the input price vector $w \in \mathbb{R}^m_{++}$ and the feasible output $y \in \mathbb{R}^s_{++}$ be given. The interval $I \subset \mathbb{R}_{++}$ with $1 \in I$ is called as a stability interval for preserving GES if y and λy for each $\lambda \in I$ are categorized in the same class of GES.

The condition $1 \in I$ guarantees that $\{\lambda y : \lambda \in I\}$ contains y. That is, stability is considered around the given output y. In a nonconvex setting a specific status of ES may not prevail in a neighborhood of the given output vector. Instead, it may occur only in separate regions (not a unified interval). This phenomenon will become clearer through the numerical examples in the current section and via the empirical studies presented in Section 7. Because of this behavior, we sometimes use the term "stability region" instead of "stability interval", since the corresponding set in $\mathbb{R}_{++}$ is not necessarily an interval.

Consider the following two notations used in the sequel. The notation "argmin" stands for the minimizers of the considered function. Similarly, "argmax" stands for the maximizers.

$$\begin{aligned} \lambda^- :=& \quad \max \operatorname{argmin} \Big\{ RAC(y, w|V)(\lambda) : \ \lambda \in (0, 1] \Big\}, \\ \lambda^+ :=& \quad \min \operatorname{argmin} \Big\{ RAC(y, w|V)(\lambda) : \ \lambda \geq 1 \Big\}. \end{aligned} \tag{3}$$

Algorithm 1: GES Classification

Input: Input and output data; feasible output $y \in \mathbb{R}^s_{++}$; input price vector $w \in \mathbb{R}^m_{++}$
Set $C(y,w|V), C(y,w|ND), C(y,w|NI) := +\infty$
for $j = 1$ to n **do**
    Compute $\mu_j := \max\left\{\frac{y_r}{y_{rj}} : r = 1, 2, \ldots, s\right\}$
    If $\mu_j \leq 1$, then
        $C(y,w|V) := \min\{C(y,w|V), wx_j\}$
        $C(y,w|NI) := \min\{C(y,w|NI), \mu_j wx_j\}$
    If $\mu_j \geq 1$, then
        $C(y,w|ND) := \min\{C(y,w|ND), \mu_j wx_j\}$
end for
If $C(y,w|NI) = C(y,w|ND) = C(y,w|V)$, then G-CES prevails
If $C(y,w|NI) = C(y,w|ND) < C(y,w|V)$, then G-SCES prevails
If $C(y,w|NI) < C(y,w|ND) = C(y,w|V)$, then G-IES prevails
If $C(y,w|NI) < C(y,w|ND) < C(y,w|V)$, then G-SIES prevails
If $C(y,w|ND) < C(y,w|NI) = C(y,w|V)$, then G-DES prevails
If $C(y,w|ND) < C(y,w|NI) < C(y,w|V)$, then G-SDES prevails
Output: GES class at y

Indeed, λ^- is the largest optimal solution of RAC_-, while λ^+ is the smallest optimal solution of RAC_+. The following corollary is a consequence of previous results.

Corollary 1. Let the input price vector $w \in \mathbb{R}^m_{++}$ and the feasible output $y \in \mathbb{R}^s_{++}$ be given.

(i) G-CES prevails if and only if $RAC_- = RAC_+$ and $\lambda^- = \lambda^+ = 1$.

(ii) G-SCES prevails if and only if $RAC_- = RAC_+$ and $\lambda^- < 1 < \lambda^+$.

(iii) G-IES prevails if and only if $RAC_- > RAC_+$ and $\lambda^- = 1$.

(iv) G-SIES prevails if and only if $RAC_- > RAC_+$ and $\lambda^- < 1 < \lambda^+$.

(v) G-DES prevails if and only if $RAC_- < RAC_+$ and $\lambda^+ = 1$.

(vi) G-SDES prevails if and only if $RAC_- < RAC_+$ and $\lambda^- < 1 < \lambda^+$.

6.1 Stability: Constant Case

Part (i) of Theorem 2 below provides a stability interval for G-SCES in terms of λ^- and λ^+. The matter is different for G-CES. Part (ii) of Theorem 2 shows that there is no stability interval for the G-CES case. This can be understood from Figure 1 as well. In this figure, G-CES prevails at $y = 10$ observed for DMU D, while in each neighborhood of $y = 10$ there is some output level at which G-CES does not prevail. Indeed, sufficiently small decreases from $y = 10$ lead to an intersection with the G-IES area, and sufficiently small increases from $y = 10$ position us in the G-SCES area.

Theorem 2. Let the feasible output level $y \in \mathbb{R}^s_{++}$ and the input price vector $w \in \mathbb{R}^m_{++}$ be given.

(i) If G-SCES prevails at y, then (λ^-, λ^+) is a stability interval.

(ii) If G-CES prevails at y, the only stability interval is $I = \{1\}$.

In Example 1, for $y = 12$ with G-SCES status, according to the calculations given in Table 2 and Proposition 1, we have $\lambda^- = \frac{10}{12}$ and $\lambda^+ = \frac{20}{12}$. So, the stability interval is $(\frac{10}{12}, \frac{20}{12})$. It means for all output levels within the interval $(10, 20)$, GES status is G-SCES. This is verified from Figure 1 as well. Furthermore, in Figure 1, G-CES prevails at $y = 10$ without any non-trivial stability interval.

6.2 Stability: General Case

After determining the GES class, we need to design algorithms for getting stability intervals preserving the GES. For the G-SCES case, due to Theorem 2, it is sufficient to obtain the values of λ^- and λ^+ defined in (3). For obtaining λ^-, due to $RAC_- = C(y, w|ND)$ (see the proof of Theorem 1) and Proposition 1, we solve (2-ND) in an enumerative manner, and then obtain the maximum $\frac{1}{\mu_j}$ among its optimal solutions. Analogously, for obtaining λ^+, due to $RAC_+ = C(y, w|NI)$ (see the proof of Theorem 1) and Proposition 1, we solve (2-NI) in an enumeration manner, and then obtain the minimum $\frac{1}{\mu_j}$ among its optimal solutions. These lead us to Algorithm 2. In this algorithm, we use two dummy sets storing the optimal solutions of the corresponding problems.

Notice that, by Theorem 2, when G-CES prevails, the only stability interval is $I = \{1\}$.

Computational complexity. The initialization step of Algorithm 2 requires $O(1)$ time. The main loop is executed n times. For each observation j, the scaling factor μ_j is computed in $O(s)$ time, while the cost term wx_j requires $O(m)$ arithmetic operations. The subsequent updates of the quantities $C(y, w|V)$, $C(y, w|NI)$, and $C(y, w|ND)$ require constant time. The sets P and Q store values associated with some of the observations. In the worst case, each set may contain up to n elements; hence, the final computation of $\lambda^- = \max P$ and $\lambda^+ = \min Q$ requires $O(|P|+|Q|) \le O(n)$ time. Therefore, the overall computational complexity of the algorithm is $O\big(n(m+s)\big)$. Moreover, since in practice, $n \ge 3(m+s)$, it follows that $m + s \le n/3$, and thus the complexity is bounded above by $O(n^2)$. However, the tighter expression $O(n(m+s))$ is retained to reflect the dependence on the numbers of inputs and outputs. Throughout the complexity analysis, constant factors are ignored in Big-O notation. In summary, Algorithm 1 runs in polynomial time with respect to the number of observations.

Now, we work on stability for G-IES, G-SIES, G-DES, and G-SDES. To this end, we get closed formulas for three functions $C(\lambda y, w|V)$, $C(\lambda y, w|NI)$, and $C(\lambda y, w|ND)$ by applying enumeration procedures. Then, stability intervals/regions are derived by comparing these functions and applying Theorem 1.

The closed formulas for the three functions $C(\lambda y, w|V)$, $C(\lambda y, w|NI)$, and $C(\lambda y, w|ND)$ are described in Theorem 3. We refer to the proof of this theorem in the Appendix A for a better understanding of these formulas.

Theorem 3. Let the feasible output level $y \in \mathbb{R}^s_{++}$ and the input price vector $w \in \mathbb{R}^m_{++}$ be given. By setting $\theta_k := \min_r \frac{y_{rk}}{y_r}$ for $k = 1, 2, \ldots, n$ assume $\theta_0 := 0 \le \theta_1 \le \theta_2 \le \ldots \le \theta_n$. Then:

(i) For each $i \in \{1, 2, \ldots, n\}$ and all $\lambda \in (\theta_{i-1}, \theta_i]$, we have $C(\lambda y, w \mid V) = \min\limits_{k=i,i+1,\ldots,n} wx_k$.

Algorithm 2: Stability interval for G-SCES

Input: Input and output data, feasible output vector $y \in \mathbb{R}^s_{++}$, and input price vector $w \in \mathbb{R}^m_{++}$
Set $C(y,w|V), C(y,w|ND), C(y,w|NI) := +\infty$ and $P, Q := \emptyset$
for $j = 1$ to n **do**
    Compute $\mu_j := \max\left\{\frac{y_r}{y_{rj}} : r = 1,2,\ldots,s\right\}$
    if $\mu_j \leq 1$, **then**
        $C(y,w|V) := \min\{C(y,w|V), wx_j\}$
        If $\mu_j wx_j < C(y,w|NI)$, then
            $C(y,w|NI) := \mu_j wx_j$ and $P := \{\frac{1}{\mu_j}\}$
        If $\mu_j wx_j = C(y,w|NI)$, then
            $C(y,w|NI) := \mu_j wx_j$ and $P := P \cup \{\frac{1}{\mu_j}\}$
    end if
    if $\mu_j \geq 1$, **then**
        If $\mu_j wx_j < C(y,w|ND)$, then
            $C(y,w|ND) := \mu_j wx_j$ and $Q := \{\frac{1}{\mu_j}\}$
        If $\mu_j wx_j = C(y,w|ND)$, then
            $C(y,w|ND) := \mu_j wx_j$ and $Q := Q \cup \{\frac{1}{\mu_j}\}$
    end if
end for
Set $\lambda^- := \max P$ and $\lambda^+ := \min Q$
If $C(y,w|NI) = C(y,w|ND) = C(y,w|V)$, then G-CES prevails
If $C(y,w|NI) = C(y,w|ND) < C(y,w|V)$, then G-SCES prevails with stability interval $I = (\lambda^-, \lambda^+)$
Output: A stability interval for G-SCES

(ii) For each $i \in \{1,2,\ldots,n\}$ and all $\lambda \in (\theta_{i-1}, \theta_i]$, we have $C(\lambda y, w \mid NI) = \min\limits_{k=i,i+1,\ldots,n} \dfrac{\lambda wx_k}{\theta_k}$.

(iii) For all $\lambda \in (\theta_0, \theta_1]$, we have $C(\lambda y, w \mid ND) = \min\{wx_k : \ k = 1,\ldots,n\}$. Furthermore, for each $i \in \{2,\ldots,n\}$ and all $\lambda \in (\theta_{i-1}, \theta_i]$, we have $C(\lambda y, w \mid ND) = \min\{\frac{\lambda wx_k}{\theta_k},\ k = 1,2,\ldots,i-1,\ \&\ wx_k,\ k = i, i+1,\ldots,n\}$.

After getting the closed formulas summarized in Theorem 3, by comparing these functions and applying Theorem 1, stability intervals are obtained.

For a better understanding of the above-mentioned procedure, we apply it on some output values in Example 1. Hereafter, given the set $\mathcal{A}$ and the scalar α, by $\alpha\mathcal{A}$ we mean the set $\{\alpha x : \ x \in \mathcal{A}\}$.

■ Consider $y = 3$ with G-IES status. We have calculated θ_k, wx_k, and $\frac{wx_k}{\theta_k}$ values for $y = 3$ as reported in the left part of Table 3, and $C(\lambda y, w|V)$, $C(\lambda y, w|NI)$, and $C(\lambda y, w|ND)$ as described in the right side of this table. The fifth column of Table 3 (containing π_k values) is defined and used in Subsection 6.3 (Example 2).

Now, due to Table 3, $C(\lambda y, w|NI) < C(\lambda y, w|ND) = C(\lambda y, w|V)$ if λ belongs to $\mathcal{A}_3 := (0, \frac{1}{3}] \cup \{1\} \cup [\frac{7}{3}, \frac{10}{3}]$. It means G-IES prevails at any $y \in 3\mathcal{A}_3 = (0,1] \cup \{3\} \cup [7,10]$, which is compatible with Figure 1. In summary, $\mathcal{A}_3$ is a stability region for G-IES at $y = 3$.

■ Consider $y = 5$ with G-SIES status. The values of θ_k, wx_k, and $\frac{wx_k}{\theta_k}$ for $y = 5$ are as reported in the left part of Table 4, and $C(\lambda y, w|V)$, $C(\lambda y, w|NI)$, and $C(\lambda y, w|ND)$ are as described in the right part of this table. The two last columns of Table 4 (i.e., π_k and h_k) are defined and used in Subsection 6.3 (Example 3).

Now, due to Theorem 1, as G-SIES prevails at $y = 5$, for getting the stability region, we should

k	θ_k	wx_k	$\gamma_k := \frac{wx_k}{\theta_k}$	π_k	λ-interval	$C(\lambda y, w\|V)$	$C(\lambda y, w\|NI)$	$C(\lambda y, w\|ND)$
A	$\frac{1}{3}$	1	3	1	$0 < \lambda \leq \frac{1}{3}$	1	$\frac{21}{10}\lambda$	1
B	1	3	3	3	$\frac{1}{3} < \lambda \leq 1$	3	$\frac{21}{10}\lambda$	$\min\{3, 3\lambda\} = 3\lambda$
C	$\frac{5}{3}$	6	$\frac{18}{5}$	6	$1 < \lambda \leq \frac{5}{3}$	6	$\frac{21}{10}\lambda$	$\min\{6, 3\lambda, 3\lambda\} = 3\lambda$
D	$\frac{10}{3}$	7	$\frac{21}{10}$	7	$\frac{5}{3} < \lambda \leq \frac{10}{3}$	7	$\frac{21}{10}\lambda$	$\min\{7, 3\lambda, 3\lambda, \frac{18}{5}\lambda\} = \begin{cases} 3\lambda, & \frac{5}{3} < \lambda \leq \frac{7}{3} \\ 7, & \frac{7}{3} < \lambda \leq \frac{10}{3} \end{cases}$
E	4	11	$\frac{11}{4}$	11	$\frac{10}{3} < \lambda \leq 4$	11	$\frac{21}{10}\lambda$	$\min\{11, 3\lambda, 3\lambda, \frac{18}{5}\lambda, \frac{21}{10}\lambda\} = \frac{21}{10}\lambda$
F	$\frac{20}{3}$	14	$\frac{42}{20}$	14	$4 < \lambda \leq \frac{20}{3}$	14	$\frac{21}{10}\lambda$	$\min\{14, 3\lambda, 3\lambda, \frac{18}{5}\lambda, \frac{21}{10}\lambda, \frac{11}{4}\lambda\} = \frac{21}{10}\lambda$

Table 3: Values of θ_k, wx_k, and $\frac{wx_k}{\theta_k}$ and $C(\lambda y, w|V)$, $C(\lambda y, w|NI)$, and $C(\lambda y, w|ND)$ for $y = 3$.

k	θ_k	wx_k	$\gamma_k := \frac{wx_k}{\theta_k}$	π_k	h_k	λ-interval	$C(\lambda y, w\|V)$	$C(\lambda y, w\|NI)$	$C(\lambda y, w\|ND)$
A	$\frac{1}{5}$	1	5	1	$\frac{7}{2}$	$0 < \lambda \leq \frac{1}{5}$	1	$\frac{7}{2}\lambda$	1
B	$\frac{3}{5}$	3	5	3	$\frac{7}{2}$	$\frac{1}{5} < \lambda \leq \frac{3}{5}$	3	$\frac{7}{2}\lambda$	$\min\{3, 5\lambda\} = 5\lambda$
C	1	6	6	6	$\frac{7}{2}$	$\frac{3}{5} < \lambda \leq 1$	6	$\frac{7}{2}\lambda$	$\min\{6, 5\lambda, 5\lambda\} = 5\lambda$
D	2	7	$\frac{7}{2}$	7	$\frac{7}{2}$	$1 < \lambda \leq 2$	7	$\frac{7}{2}\lambda$	$\min\{7, 5\lambda, 5\lambda, 6\lambda\} = \begin{cases} 5\lambda, & 1 < \lambda \leq \frac{7}{5} \\ 7, & \frac{7}{5} < \lambda \leq 2 \end{cases}$
E	$\frac{12}{5}$	11	$\frac{55}{12}$	11	$\frac{7}{2}$	$2 < \lambda \leq \frac{12}{5}$	11	$\frac{7}{2}\lambda$	$\min\{11, 5\lambda, 5\lambda, 6\lambda, \frac{7}{2}\lambda\} = \frac{7}{2}\lambda$
F	4	14	$\frac{14}{4}$	14	$\frac{7}{2}$	$\frac{12}{5} < \lambda \leq 4$	14	$\frac{7}{2}\lambda$	$\min\{14, 5\lambda, 5\lambda, 6\lambda, \frac{7}{2}\lambda, \frac{55}{12}\lambda\} = \frac{7}{2}\lambda$

Table 4: Values of θ_k, wx_k, and $\frac{wx_k}{\theta_k}$ and $C(\lambda y, w|V)$, $C(\lambda y, w|NI)$, and $C(\lambda y, w|ND)$ for $y = 5$.

determine the λ values in which $C(\lambda y, w|NI) < C(\lambda y, w|ND) < C(\lambda y, w|V)$. According to Table 4, it holds if λ belongs to $\mathcal{A}_5 := (\frac{1}{5}, \frac{3}{5}) \cup (\frac{3}{5}, \frac{7}{5})$. It means G-SIES prevails at any $y \in 5\mathcal{A}_5 = (1, 3) \cup (3, 7)$ (which is compatible with Figure 1). In summary, $\mathcal{A}_5$ is a stability region for G-SIES at $y = 5$, and $(\frac{3}{5}, \frac{7}{5})$ is a stability interval.

Although Theorem 2 provides a stability interval for G-SCES in terms of λ^- and λ^+, a stability region can also be derived using the above analysis. In the G-CES case, the only stability interval is $I = 1$ (Theorem 2), but a corresponding stability region can still be obtained through the same analysis.

The following examples illustrate these points.

■ Consider $y = 12$ with G-SCES status. The values of θ_k, wx_k, $\frac{wx_k}{\theta_k}$ and also $C(\lambda y, w|V)$, $C(\lambda y, w|NI)$, and $C(\lambda y, w|ND)$ are as described in Table 5.

Now, due to Theorem 1, as G-SCES prevails at $y = 12$, for getting the stability region, we should determine the λ values in which $C(\lambda y, w|NI) = C(\lambda y, w|ND) < C(\lambda y, w|V)$. According to Table 5, it holds if λ belongs to $\mathcal{A}_{12} = (\frac{10}{12}, \frac{20}{12})$. It means G-SIES prevails at any $y \in 12\mathcal{A}_{12} = (10, 20)$ (which is compatible with Figure 1 as well as Theorem 2). In summary, $\mathcal{A}_{12}$ is a stability region and stability interval for G-SCES at $y = 12$.

k	θ_k	wx_k	$\frac{wx_k}{\theta_k}$	λ-interval	$C(\lambda y, w\|V)$	$C(\lambda y, w\|NI)$	$C(\lambda y, w\|ND)$
A	$\frac{1}{12}$	1	12	$0 < \lambda \leq \frac{1}{12}$	1	$\frac{42}{5}\lambda$	1
B	$\frac{1}{4}$	3	12	$\frac{1}{12} < \lambda \leq \frac{1}{4}$	3	$\frac{42}{5}\lambda$	$\min\{3, 12\lambda\} = 12\lambda$
C	$\frac{5}{12}$	6	$\frac{72}{5}$	$\frac{1}{4} < \lambda \leq \frac{5}{12}$	6	$\frac{42}{5}\lambda$	$\min\{6, 12\lambda, 12\lambda\} = 12\lambda$
D	$\frac{10}{12}$	7	$\frac{42}{5}$	$\frac{5}{12} < \lambda \leq \frac{10}{12}$	7	$\frac{42}{5}\lambda$	$\min\{7, 12\lambda, 12\lambda, \frac{72}{5}\lambda\} = \begin{cases} 12\lambda, & \frac{5}{12} < \lambda \leq \frac{7}{12} \\ 7, & \frac{7}{12} < \lambda \leq \frac{10}{12} \end{cases}$
E	1	11	11	$\frac{10}{12} < \lambda \leq 1$	11	$\frac{42}{5}\lambda$	$\min\{11, 12\lambda, 12\lambda, \frac{72}{5}\lambda, \frac{42}{5}\lambda\} = \frac{42}{5}\lambda$
F	$\frac{20}{12}$	14	$\frac{42}{5}$	$1 < \lambda \leq \frac{20}{12}$	14	$\frac{42}{5}\lambda$	$\min\{14, 12\lambda, 12\lambda, \frac{72}{5}\lambda, \frac{42}{5}\lambda, 11\lambda\} = \frac{42}{5}\lambda$

Table 5: Values of θ_k, wx_k, and $\frac{wx_k}{\theta_k}$ and $C(\lambda y, w|V)$, $C(\lambda y, w|NI)$, and $C(\lambda y, w|ND)$ for $y = 12$.

■ Consider $y = 10$ with G-CES status. The values of θ_k, wx_k, and $\frac{wx_k}{\theta_k}$ alongside $C(\lambda y, w|V)$, $C(\lambda y, w|NI)$, and $C(\lambda y, w|ND)$ for $y = 10$ are as described in Table 6.

As G-CES prevails at $y = 10$, invoking Theorem 1, for getting the stability region, we should

k	θ_k	wx_k	$\frac{wx_k}{\theta_k}$	λ-interval	$C(\lambda y, w\|V)$	$C(\lambda y, w\|NI)$	$C(\lambda y, w\|ND)$
A	$\frac{1}{10}$	1	10	$0 < \lambda \leq \frac{1}{10}$	1	7λ	1
B	$\frac{3}{10}$	3	10	$\frac{1}{10} < \lambda \leq \frac{3}{10}$	3	7λ	$\min\{3, 10\lambda\} = 10\lambda$
C	$\frac{1}{2}$	6	12	$\frac{3}{10} < \lambda \leq \frac{1}{2}$	6	7λ	$\min\{6, 10\lambda, 10\lambda\} = 10\lambda$
D	1	7	7	$\frac{1}{2} < \lambda \leq 1$	7	7λ	$\min\{7, 10\lambda, 10\lambda, 12\lambda\} = \begin{cases} 10\lambda, & \frac{1}{2} < \lambda \leq \frac{7}{10} \\ 7, & \frac{7}{10} < \lambda \leq 1 \end{cases}$
E	$\frac{12}{10}$	11	$\frac{55}{6}$	$1 < \lambda \leq \frac{12}{10}$	11	7λ	$\min\{11, 10\lambda, 10\lambda, 12\lambda, 7\lambda\} = 7\lambda$
F	2	14	7	$\frac{12}{10} < \lambda \leq 2$	14	7λ	$\min\{14, 10\lambda, 10\lambda, 12\lambda, 7\lambda, \frac{55}{6}\lambda\} = 7\lambda$

Table 6: Values of θ_k, wx_k, and $\frac{wx_k}{\theta_k}$ and $C(\lambda y, w|V)$, $C(\lambda y, w|NI)$, and $C(\lambda y, w|ND)$ for $y = 10$.

determine the λ values in which $C(\lambda y, w|NI) = C(\lambda y, w|ND) = C(\lambda y, w|V)$. According to Table 6, it holds if λ belongs to $\mathcal{A}_{10} = \{1, 2\}$. It means G-CES prevails at any $y \in 10\mathcal{A}_{10} = \{10, 20\}$. In summary, $\mathcal{A}_{10}$ is a stability region and $\{1\}$ is the stability interval for G-CES at $y = 10$ (compatible with Theorem 2).

■ As the final examination of this section, we investigate $y = 8$ as a non-extreme case, with θ_k, wx_k, and $\frac{wx_k}{\theta_k}$ quantities and $C(\lambda y, w|V)$, $C(\lambda y, w|NI)$, and $C(\lambda y, w|ND)$ functions described in Table 7.

By setting $\lambda = 1$ in Table 7, for $y = 8$, we have $C(\lambda y, w|NI) = \frac{28}{5} < 7 = C(\lambda y, w|ND) = C(\lambda y, w|V)$. Thus, by Theorem 1, G-IES prevails at $y = 8$. Furthermore, according to Table 7 , $C(\lambda y, w|NI) < C(\lambda y, w|ND) = C(\lambda y, w|V)$ if λ belongs to $\mathcal{A}_8 = (0, \frac{1}{8}] \cup \{\frac{3}{8}\} \cup [\frac{7}{8}, \frac{10}{8})$. It means G-IES prevails at any $y \in 8\mathcal{A}_8 = (0, 1] \cup \{3\} \cup [7, 10]$, which is compatible with Figure 1. In summary, $\mathcal{A}_8$ is a stability region for G-IES at $y = 8$, and $[\frac{7}{8}, \frac{10}{8}]$ is a stability interval.

k	θ_k	wx_k	$\frac{wx_k}{\theta_k}$	λ-interval	$C(\lambda y, w\|V)$	$C(\lambda y, w\|NI)$	$C(\lambda y, w\|ND)$
A	$\frac{1}{8}$	1	8	$0 < \lambda \leq \frac{1}{8}$	1	$\frac{28}{5}\lambda$	1
B	$\frac{3}{8}$	3	8	$\frac{1}{8} < \lambda \leq \frac{3}{8}$	3	$\frac{28}{5}\lambda$	$\min\{3, 8\lambda\} = 8\lambda$
C	$\frac{5}{8}$	6	$\frac{48}{5}$	$\frac{3}{8} < \lambda \leq \frac{5}{8}$	6	$\frac{28}{5}\lambda$	$\min\{6, 8\lambda, 8\lambda\} = 8\lambda$
D	$\frac{10}{8}$	7	$\frac{28}{5}$	$\frac{5}{8} < \lambda \leq \frac{10}{8}$	7	$\frac{28}{5}\lambda$	$\min\{7, 8\lambda, 8\lambda, \frac{48}{5}\lambda\} = \begin{cases} 8\lambda, & \frac{5}{8} < \lambda \leq \frac{7}{8} \\ 7, & \frac{7}{8} < \lambda \leq \frac{10}{8} \end{cases}$
E	$\frac{12}{8}$	11	$\frac{22}{3}$	$\frac{10}{8} < \lambda \leq \frac{12}{8}$	11	$\frac{28}{5}\lambda$	$\min\{11, 8\lambda, 8\lambda, \frac{48}{5}\lambda, 7\lambda\} = 7\lambda$
F	$\frac{20}{8}$	14	$\frac{28}{5}$	$\frac{12}{8} < \lambda \leq \frac{20}{8}$	14	$\frac{28}{5}\lambda$	$\min\{14, 8\lambda, 8\lambda, \frac{48}{5}\lambda, 7\lambda, \frac{22}{3}\lambda\} = 7\lambda$

Table 7: Values of θ_k, wx_k, and $\frac{wx_k}{\theta_k}$ and $C(\lambda y, w|V)$, $C(\lambda y, w|NI)$, and $C(\lambda y, w|ND)$ for $y = 8$.

6.3 Stability: Enumeration Results

The following results provide sufficient conditions that lead to stability regions for GES, thanks to the closed-form formulas listed in Theorem 3. For the sake of simplicity, we use the following notations hereafter. All these quantities can be easily obtained from the given data (input-output data and the weight vector w) by simple enumeration methods:

$$\begin{aligned}
&\theta_k := \min_r \frac{y_{rk}}{y_r} \text{ for } k = 1, 2, \dots, n, \\
&\pi_k := \min\{wx_k, wx_{k+1}, \dots, wx_n\}, \\
&\gamma_k := \tfrac{wx_k}{\theta_k} \quad \text{for} \quad k = 1, 2, \dots, n, \\
&h_k := \min\{\gamma_k, \gamma_{k+1}, \dots, \gamma_n\}.
\end{aligned}$$

Sort DMUs such that $\theta_1 \leq \theta_2 \leq \ldots \leq \theta_n$. Theorem 4 provides stability regions for G-IES. Notice that, we ignore the empty intervals in the following two results.

Theorem 4. Let the feasible output level $y \in \mathbb{R}^s_{++}$ and the input price vector $w \in \mathbb{R}^m_{++}$ be given.

(i) G-IES prevails at λy for each $\lambda \in (0, \theta_1)$.

(ii) G-IES prevails at λy for each $\lambda \in \Big(\max\{\theta_1, \frac{\pi_2}{\gamma_1}\}, \theta_2\Big)$.

(iii) G-IES prevails at λy for each $\lambda \in \Big(\max\{\theta_2, \frac{\pi_3}{\gamma_1}, \frac{\pi_3}{\gamma_2}\}, \theta_3\Big)$.

(iv) In general, for $t \in \{2, 3, \dots, n\}$, G-IES prevails at λy for each $\lambda \in \Big(\max\{\theta_{t-1}, \frac{\pi_t}{\gamma_1}, \frac{\pi_t}{\gamma_2}, \dots, \frac{\pi_t}{\gamma_{t-1}}\}, \theta_t\Big)$.

Example 2. In Example 1, consider $y = 3$. The values of θ_k, γ_k, and π_k are $(\frac{1}{3}, 1, \frac{5}{3}, \frac{10}{3}, 4, \frac{20}{3})$, $(3, 3, \frac{18}{5}, \frac{21}{10}, \frac{11}{4}, \frac{42}{20})$, and $(1, 3, 6, 7, 11, 14)$, respectively. Since $\theta_1 = \frac{1}{3}$, due to Theorem 4, G-IES prevails at λy for each $\lambda \in (0, \frac{1}{3})$. The only nonempty interval among those addressed in Theorem 4, is:

$$\Big(\max\{\theta_3, \frac{\pi_4}{\gamma_1}, \frac{\pi_4}{\gamma_2}, \frac{\pi_4}{\gamma_3}\}, \theta_4\Big) = \Big(\max\{\frac{5}{3}, \frac{7}{3}, \frac{35}{18}\}, \frac{10}{3}\Big) = (\frac{7}{3}, \frac{10}{3}).$$

Hence, by Theorem 4, G-IES prevails at λy for each $\lambda \in (\frac{7}{3}, \frac{10}{3})$. In summary, G-IES prevails at each $y \in (0,1) \cup \{3\} \cup (7,10)$, compatible with Figure 1.

Theorem 5 concentrates on the G-SIES case.

Theorem 5. Let the feasible output level $y \in \mathbb{R}^s_{++}$ and the input price vector $w \in \mathbb{R}^m_{++}$ be given.

(i) If $h_2 < \gamma_1$, then G-SIES prevails at λy for each $\lambda \in \left(\theta_1, \min\{\theta_2, \frac{\pi_2}{\gamma_1}\}\right)$.

(ii) If $h_3 < \min\{\gamma_1, \gamma_2\}$, then G-SIES prevails at λy for each

$$\lambda \in \left(\theta_2, \min\{\theta_3, \max\{\frac{\pi_3}{\gamma_1}, \frac{\pi_3}{\gamma_2}\}\}\right).$$

(iii) In general, let $t \in \{2, 3, \dots, n\}$. If $h_t < \min\{\gamma_1, \dots, \gamma_{t-1}\}$, then G-SIES prevails at λy for each

$$\lambda \in \left(\theta_{t-1}, \min\{\theta_t, \max\{\frac{\pi_t}{\gamma_1}, \dots, \frac{\pi_t}{\gamma_{t-1}}\}\}\right).$$

Example 3. In Example 1, consider $y = 5$. The values of θ_k, π_k, and γ_k are $(\frac{1}{5}, \frac{3}{5}, 1, 2, \frac{12}{5}, 4)$, $(1, 3, 6, 7, 11, 14)$, and $(5, 5, 6, \frac{7}{2}, \frac{55}{12}, \frac{14}{4})$, respectively. Furthermore, all h_k values are equal to $\frac{7}{2}$. There are three feasible cases as follows.
(1): $h_2 = \frac{7}{2} < \gamma_1 = 5$ and the interval is $(\frac{1}{5}, \frac{3}{5})$.
(2): $h_3 = \frac{7}{2} < \min\{\gamma_1, \gamma_2\} = 5$ and the interval is $(\frac{3}{5}, 1)$.
(3): $h_4 = \frac{7}{2} < \min\{\gamma_1, \gamma_2, \gamma_3\} = 5$ and the interval is $(1, \frac{7}{5})$.
Hence, by Theorem 5, G-SIES prevails at λy for each $\lambda \in (\frac{1}{5}, \frac{3}{5}) \cup (\frac{3}{5}, 1) \cup (1, \frac{7}{5})$. In summary, G-SIES prevails at each $y \in (1,3) \cup (3,7)$, compatible with Figure 1.

An interesting result concerns the G-CES case. In Theorem 2, we established that if G-CES prevails at y, then the only stability interval is $I = \{1\}$. This conclusion can also be inferred from the closed-form formulas presented in Table 3. If we add the output level y to the observed ones, then one of the θ_k values becomes equal to one. From Theorem 3, we see that $C(\lambda y, w \mid NI) < C(\lambda y, w \mid V)$ for any λ belonging to either of the open intervals (θ_{i-1}, θ_i). Hence, G-CES does not prevail at λy for any λ in these open intervals. This implies that the only stability interval is $I = \{1\}$.

The aforementioned interesting property is also valid for the G-DES status. Since $C(\lambda y, w \mid NI) < C(\lambda y, w \mid V)$ for any λ belonging to either of the open intervals (θ_{i-1}, θ_i), due to Theorem 1, the only stability interval for the G-DES case is $I = \{1\}$. We state this formally in the following Theorem 6.

Theorem 6. Let the feasible output level $y \in \mathbb{R}^s_{++}$ and the input price vector $w \in \mathbb{R}^m_{++}$ be given. If G-DES prevails at y, the only stability interval is $I = \{1\}$.

Theorem 7 below provides a stability region for G-SDES. Here, we use a new notation, $\nu_k := \min\{\gamma_1, \gamma_2, \dots, \gamma_k\}$.

Theorem 7. Let the feasible output level $y \in \mathbb{R}^s_{++}$ and the input price vector $w \in \mathbb{R}^m_{++}$ be given.

(i) G-SDES prevails at λy for each $\lambda \in (\theta_{n-1}, \min\{\theta_n, \frac{wx_n}{\nu_{n-1}}\})$ provided that $\nu_{n-1} < \frac{wx_n}{\theta_n}$.

(ii) G-SDES prevails at λy for each $\lambda \in (\theta_{n-2}, \min\{\theta_{n-1}, \frac{\min\{wx_{n-1}, wx_n\}}{\nu_{n-2}}\})$ provided that $\nu_{n-2} < \min\{\frac{wx_{n-1}}{\theta_{n-1}}, \frac{wx_n}{\theta_n}\}$.

(iii) In general, let $t \in \{2, 3, \ldots, n\}$. G-SDES prevails at λy for each $\lambda \in (\theta_{t-1}, \min\{\theta_t, \frac{\min\{wx_t, \ldots, wx_n\}}{\nu_{t-1}}\})$ provided that $\nu_{t-1} < \min\{\frac{wx_t}{\theta_t}, \ldots, \frac{wx_n}{\theta_n}\}$.

Finally, notice that the RAC function can also be defined in a non-proportional output-mix adjustment framework. Correspondingly, stability intervals can be derived within this same non-proportional output-mix framework: see Appendix C for more details.

7 Empirical Illustration

We present the outcomes of implementing our results on real-world data sets. We conduct experiments on two data sets from *electricity generation* and *electricity distribution* sectors. Kerstens & Zhao (2025) forcefully argue from the engineering literature that there is good reason to expect the electricity sector to have a nonconvex technology. These authors also provide statistical tests rejecting convexity of the cost function in the outputs for the first data set.

The first experiment is conducted on a data set extracted from Kumbhakar & Tsionas (2011) in the electricity generation sector.[5] Ignoring the time dimension, the data set contains *1065 DMUs* under assessment, with three inputs and one output. Since the data set is large, we report only the information of a few units in Table 8. We do not use the names of the utilities. Instead, we assign a numeric identifier to each (the second row of the table). The first three rows after the DMU identifier represent the input prices, while the next three rows provide the corresponding input quantities. The seventh row contains the output values. Following these, rows eight through ten display different *nonconvex* cost measures: non-increasing ($C(y, w|NI)$ abbreviated as C_{NI}), non-decreasing ($C(y, w|ND)$ abbreviated as C_{ND}), and variable ($C(y, w|V)$ abbreviated as C_V). The "NC-Class" row indicates the classification of DMUs according to GES, followed by three rows presenting *convex* optimal cost values ($C^c(y, w|NI)$, $C^c(y, w|ND)$, and $C^c(y, w|V)$, abbreviated as C^c_{NI}, C^c_{ND}, and C^c_V, respectively: see (4)). The final row, "C-Class," categorizes the class of each DMU according to *convex ES*.

In this experiment, we find one unit with *G-CES* status, 85 units with *G-IES*, 301 with *G-SIES*, 4 with *G-DES*, and 674 with *G-SDES* status. G-SCES does not happen for this data set. However, it may happen in nonconvex technologies as seen in Example 1. Figure 5 illustrates the distribution of these units across the various GES classes. Notably, more than *91%* of the units fall into the new classes introduced in the present paper, G-SIES and G-SDES.

[5]Data are available from the Journal of Applied Econometrics repository: http://qed.econ.queensu.ca/jae/2011-v26.2/kumbhakar-tsionas/.

	DMU		DMU 44		DMU 61		DMU 104
DMU	1	...	44	...	61	...	104
w_1	17.39246	...	18.62509	...	32.15046	...	288.769
w_2	1.961324	...	2.203031	...	1.942966	...	1.764471
w_3	0.11806	...	0.13867	...	0.13044	...	0.12043
x_1	7556	...	4935	...	411	...	179
x_2	277817	...	149636	...	19074	...	93324
x_3	1724829	...	994063	...	72486	...	640574
y	37174620	...	21361100	...	1713360	...	10328646
C_{NI}	719998.7	...	468719.6904	...	34434.11173	...	293501.4694
C_{ND}	624081	...	398257.9146	...	47889.64333	...	293501.4694
C_V	828589	...	559414.2821	...	47889.64333	...	293501.4694
NC-Class	G-SDES	...	G-SDES	...	G-IES	...	G-CES
C^c_{NI}	714348.1	...	452454.042	...	34434.11173	...	293501.4694
C^c_{ND}	624081	...	398257.9146	...	41445.3613	...	293501.4694
C^c_V	714348.1	...	452454.042	...	41445.3613	...	293501.4694
C-Class	Conv-DES	...	Conv-DES	...	Conv-IES	...	Conv-CES

Table 8: Data and results in the electricity generation experiment (4 selected DMUs)

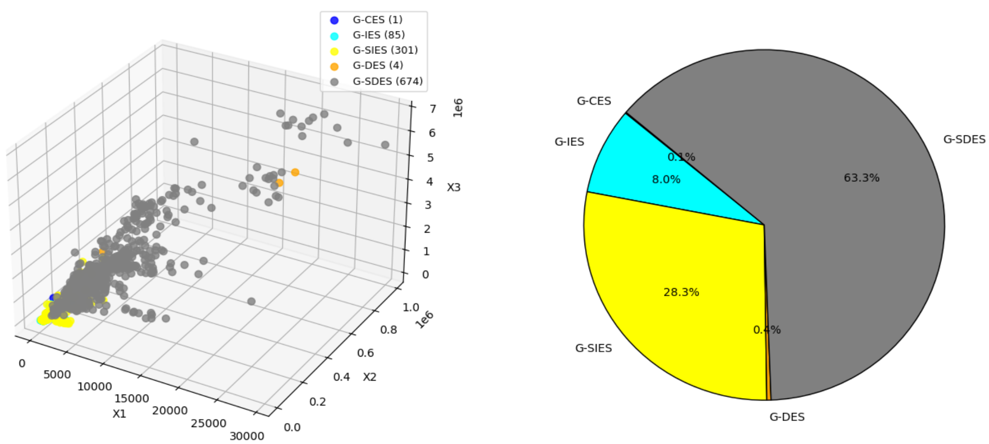

Figure 5: Distribution of DMUs by (nonconvex) GES classification in the electricity generation data set

In addition to the nonconvex setting, we have analyzed and compared the results on the convex case as well. For the convex case, we use the following classification extracted from Kerstens & Van de Woestyne (2021) by a simple improvement. In the convex case, we work with three cost models as

$$C^c(y,w|V) = \min\ wx\ s.t. \sum_{j\in J}\mu_j x_j \le x,\ \sum_{j\in J}\mu_j y_j \ge y,\ \mu\in\Lambda^{\Delta}, \tag{4}$$

with $\Lambda^V = \{\mu\in\mathbb{R}^n_+ : \sum_{j\in J}\mu_j = 1\}$, $\Lambda^{ND} = \{\mu\in\mathbb{R}^n_+ : \sum_{j\in J}\mu_j \ge 1\}$, and $\Lambda^{NI} = \{\mu\in\mathbb{R}^n_+ : \sum_{j\in J}\mu_j \le 1\}$. The superscript "$c$" refers to *convexity*. It is not difficult to see that $C^c(y,w|V) = \max\{C^c(y,w|NI), C^c(y,w|ND)\}$ (it is valid only in the convex case, and may fail in a nonconvex setting). This leads to two properties. First, the sub-constant, sub-increasing, and sub-decreasing cases do not happen in a convex technology; because $C^c(y, w|V)$ equals either $C^c(y, w|NI)$ or $C^c(y, w|ND)$. The second is that one can determine the ES class of the units in a convex setting by comparing only $C^c(y, w|NI)$ and $C^c(y, w|ND)$:

(i) In a convex technology, Constant ES (CES) prevails if and only if $C^c(y, w|NI) = C^c(y, w|ND)$.

(ii) In a convex technology, Increasing ES (IES) prevails if and only if $C^c(y, w|NI) < C^c(y, w|ND)$.

(iii) In a convex technology, Decreasing ES (DES) prevails if and only if $C^c(y, w|NI) > C^c(y, w|ND)$.

Figure 6 reports the distribution of the DMUs across the various ES classes in a *convex* setting. This experiment highlights that if the problem is *nonconvex in essence*, convex technology does not reveal the ES properly. This results in misleading conclusions about the optimal scale size and the position of an observation relative to the true flexible returns-to-scale technology. When we move to convexity, the decreasing class remains the majority and the increasing class appears as the second-largest group.

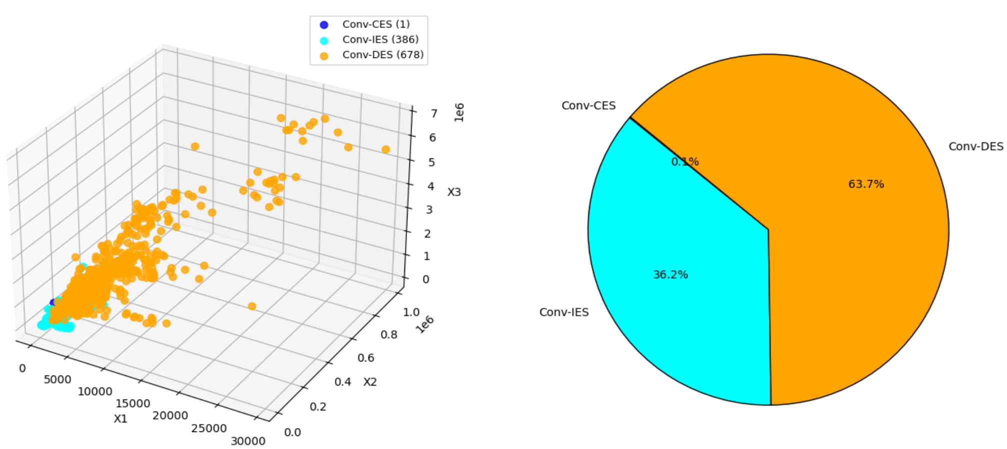

Figure 6: Distribution of DMUs by Convex-ES classification in the electricity generation data set

In our experiment with the electricity generation data set, the average running time is approximately 30 seconds for nonconvex technology and 122 seconds for the general case (convex and nonconvex).[6] Two points in the implementation should be emphasized here. First, we use the enumeration Algorithm 1 for the nonconvex case, which results in a very short runtime of around 30 seconds. If one would use nonlinear mixed-integer programming or their linearized versions instead of the enumeration approach, this runtime increases to approximately one hour (as we verified). Second, in the code, ϵ is set to a very small tolerance (10^{-12}) to handle floating-point precision

[6] All computational experiments are conducted on a laptop with an Intel Core i7 processor, 16 GB of RAM, running Windows 10 Pro, and Python 3.10 (executed via PyCharm). The python codes of all implementations can be found in this Github link: https://github.com/soleimani-damaneh/Global-Economies-of-Scale-in-Nonconvex-Technologies—2025.

limits. It prevents division by zero, relaxes strict inequality checks, and allows approximate comparisons so that tiny rounding errors do not cause misclassification. This makes the classification and optimization results more stable and reliable.

Now, we go back to the *nonconvex case* and continue our experiment with a stability analysis. To this end, we use the closed formulas derived in Subsection 6.3 (see Theorem 3). The data set examined in this experiment, contains only one output. We can get interesting observations. Assume that the cost vector w is constant among the units (usually, in practice the prices of inputs are fixed among units, however, the electricity generation data set that we are investigating addresses a more general case in which the prices change from unit to unit). Considering the output of one of the units, say y_1 of DMU$_1$, we have $y_k = \lambda_k y_1$ with $\lambda_k = \frac{y_k}{y_1}$ for $k = 2, 3, \ldots, n$. It is then sufficient to find the interval to which λ_k belongs, and by applying the closed formulas given in Theorem 3, the GES of all other units can be determined easily. This procedure works for all units when there is only one output. For multiple-output cases, it applies to the y vectors located on the ray originating from the origin and passing through the considered vector; see the next experiment. This strategy may significantly reduce the running time in the large-scale projects. For the single-output case, $\lambda_k \in (\theta_{k-1}, \theta_k]$ because $\theta_k = \min_r \dfrac{y_{rk}}{y_{r1}} = \dfrac{y_k}{y_1} = \lambda_k$.

Furthermore, in the single-output case, determining the GES class of an unobserved feasible output will be also easy due to Theorem 3. For an unobserved feasible output y, it is sufficient to set $\lambda = \frac{y}{y_1}$, then obtain the values of $C(\lambda y_1, w|V)$, $C(\lambda y_1, w|NI)$, and $C(\lambda y_1, w|ND)$, compare these values, and finally determining the GES class via Theorem 1. For multiple-output case, it is sufficient to add y to the observed outputs and then apply Algorithm 1.

Another observation relates to obtaining stability regions for GES, which applies to both single- and multiple-output cases, but is more meaningful for the multiple-output case. Therefore, we proceed to our *second experiment in the electricity distribution sector* as described in Appendix B.

8 Conclusions

In this paper, we have investigated Economies of Scale (ES), defined in terms of minimal Ray Average Cost (RAC) and optimal scale size, as a key concept in performance analysis under nonconvex production technologies. We introduced two novel notions, Global Sub-Increasing ES (G-SIES) and Global Sub-Decreasing ES (G-SDES), which are characterized by the behavior of the RAC function. The significance of these new classes in ES-based classification was revealed, and we developed technical theorems to rigorously characterize these concepts and determine the appropriate ES class for Decision Making Units (DMUs). Building upon these theoretical results, we proposed algorithms and procedures for identifying the ES classes and their associated stability regions, which remain invariant under radial perturbations of the output vectors. To demonstrate the practical applicability of our approach, we presented illustrative examples and applied the methodology to two real-world data sets from the electricity generation and electricity distribution sectors. Our findings show that, within a nonconvex framework, the newly defined notions provide valuable qualitative insights into the optimal scale size and the positioning of a DMU relative to the true flexible returns to scale technology. Moreover, we highlighted that analyses relying solely on convex production technologies may fail to accurately capture the behavior of economies of scale when the underlying technology is inherently nonconvex.

In popular cost-efficiency analysis, the cost vector is assumed to be known. Accordingly, our paper focuses on this widely studied setting. However, input prices (costs) may be unknown and/or uncertain in practice. Both cases deserve further investigation as future research directions. If the prices are unknown, then machine-learning algorithms may provide reliable tools for predicting prices from historical data (see Amini et al. (2026)). If the prices are uncertain, then interval programming (Mostafaee & Saljooghi (2010)) or robust optimization (Amini et al. (2026)) may be appropriate approaches for handling such uncertainty. Investigating these issues is a worthwhile direction for future research. Finally, we are unaware of existing studies testing the predictive power of ES classification across methods: this could be another useful topic for future work.

Acknowledgments

The last author (M. S.) acknowledges support from the Iran National Science Foundation (INSF, Grant No. 4031505). He also used ChatGPT for improving the English and assisting with LaTeX and Python debugging, without affecting the scientific content.

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Appendices: Supplementary Material

A Appendix A: Proofs of Theorems

Proof of Theorem 1:

For simplicity, we denote $\min\Big\{RAC(y,w|V)(\lambda):\ \lambda\in(0,1]\Big\}$ by RAC_-. Furthermore, we use RAC_+ for $\min\Big\{RAC(y,w|V)(\lambda):\ \lambda\geq 1\Big\}$. We have:

$$
\begin{aligned}
& RAC_- = \min\left\{\tfrac{C(\lambda y,w|V)}{\lambda}:\ \lambda\in(0,1]\right\}\\
& = \min\Big\{\tfrac{wx}{\lambda}:\ X\mu\leq x,\ Y\mu\geq\lambda y,\ x\geq 0,\ \mu=\delta\gamma,\ e\gamma=1,\ \gamma\in\{0,1\}^n,\\
& \delta=1,\ \lambda\in(0,1]\Big\}\\
& = \min\Big\{\tfrac{wx}{\lambda}:\ X\tfrac{\mu}{\lambda}\leq\tfrac{x}{\lambda},\ Y\tfrac{\mu}{\lambda}\geq y,\ x\geq 0,\ \tfrac{\mu}{\lambda}=\tfrac{1}{\lambda}\gamma,\ e\gamma=1,\\
& \ \gamma\in\{0,1\}^n,\ \lambda\in(0,1]\Big\}.
\end{aligned}
$$

Now, by setting $\mu\longleftarrow\frac{\mu}{\lambda},\ x\longleftarrow\frac{x}{\lambda}$, and $\delta\longleftarrow\frac{1}{\lambda}$, we get $\delta\geq 1$ and RAC_- equals:

$$
\min\Big\{wx:\ X\mu\leq x,\ Y\mu\geq y,\ x\geq 0,\ \mu=\delta\gamma,\ e\gamma=1,\ \gamma\in\{0,1\}^n,\ \delta\geq 1\Big\},
$$

which is equal to $C(y,w|ND)$. So,

$$
RAC_- = \min\Big\{RAC(y,w|V)(\lambda):\ \lambda\in(0,1]\Big\} = C(y,w|ND). \tag{A1}
$$

Similarly, it can be shown that

$$
RAC_+ = \min\Big\{RAC(y,w|V)(\lambda):\ \lambda\geq 1\Big\} = C(y,w|NI). \tag{A2}
$$

On the other hand, it is clear that

$$
RAC(y,w|V)(1) = C(y,w|V). \tag{A3}
$$

Now we prove the theorem.

(i): Due to Definition 3, G-CES prevails if and only if $RAC_- = RAC_+ = RAC(y,w|V)(1)$. Due to (A1)-(A3), it happens if and only if $C(y,w|ND) = C(y,w|NI) = C(y,w|V)$.

(ii): Due to Definition 3, G-SCES prevails if and only if $RAC_- = RAC_+ < RAC(y,w|V)(1)$. Due to (A1)-(A3), it happens if and only if $C(y,w|ND) = C(y,w|NI) < C(y,w|V)$.

(iii): Due to Definition 3, G-IES prevails if and only if $RAC(y,w|V)(1) = RAC_- > RAC_+$. Due to (A1)-(A3), it happens if and only if $C(y,w|V) = C(y,w|ND) > C(y,w|NI)$.

(iv): Due to Definition 3, G-SIES prevails if and only if $RAC(y,w|V)(1) > RAC_- > RAC_+$. Due to (A1)-(A3), it happens if and only if $C(y,w|V) > C(y,w|ND) > C(y,w|NI)$.

(v): Due to Definition 3, G-DES prevails if and only if $RAC_- < RAC_+ = RAC(y, w|V)(1)$. Due to (A1)-(A3), it happens if and only if $C(y, w|ND) < C(y, w|NI) = C(y, w|V)$.

(vi): Due to Definition 3, G-DES prevails if and only if $RAC_- < RAC_+ < RAC(y, w|V)(1)$. Due to (A1)-(A3), it happens if and only if $C(y, w|ND) < C(y, w|NI) < C(y, w|V)$. □

Proof of Theorem 2:

Part (i): Since G-SCES prevails at y, due to Corollary 1, $\lambda^- < 1 < \lambda^+$. Let $\lambda^* \in (\lambda^-, \lambda^+)$. According to Theorem 1, we should prove

$$C(\lambda^* y, w|NI) = C(\lambda^* y, w|ND) < C(\lambda^* y, w|V). \tag{A4}$$

Consider two problems:

$$C(y, w|ND) = \min \Big\{ wx \ : X\mu \leq x,\ Y\mu \geq y,\ \mu = \delta\gamma,\ e\gamma = 1,\ \gamma \in \{0,1\}^n,\ \delta \geq 1 \Big\} \tag{A5}$$

and

$$C(\lambda^* y, w|ND) = \min \Big\{ wx \ : X\mu \leq x,\ Y\mu \geq \lambda^* y,\ \mu = \delta\gamma,\ e\gamma = 1,\ \gamma \in \{0,1\}^n,\ \delta \geq 1 \Big\}. \tag{A6}$$

If $(x^o, \mu^o, \gamma^o, \delta^o)$ is an optimal solution of (A5), then according to Proposition 1, we have $\lambda^* > \lambda^- \geq \frac{1}{\delta^o}$, and so, $\lambda^* \delta^o \geq 1$. Hence, $(x = \lambda^* x^o, \mu = \lambda^* \mu^o, \gamma = \gamma^o, \delta = \lambda^* \delta^o)$ is a feasible solution to (A6), with the objective function value $\lambda^* w x^o$ which is equal to $\lambda^* C(y, w|ND)$. This leads in $C(\lambda^* y, w|ND) \leq \lambda^* C(y, w|ND)$.

Now assume that $(x^{oo}, \mu^{oo}, \gamma^{oo}, \delta^{oo})$ is an optimal solution of (A6). If $\frac{\delta^{oo}}{\lambda^*} \geq 1$, then $(x = \frac{x^{oo}}{\lambda^*}, \mu = \frac{\mu^{oo}}{\lambda^*}, \gamma = \gamma^{oo}, \delta = \frac{\delta^{oo}}{\lambda^*})$ is a feasible solution to (A5), with the objective function value $\frac{w x^{oo}}{\lambda^*}$ which is equal to $\frac{C(\lambda^* y, w|ND)}{\lambda^*}$. This implies, $C(y, w|ND) \leq \frac{C(\lambda^* y, w|ND)}{\lambda^*}$. If $\frac{\delta^{oo}}{\lambda^*} \leq 1$, then $(x = \frac{x^{oo}}{\lambda^*}, \mu = \frac{\mu^{oo}}{\lambda^*}, \gamma = \gamma^{oo}, \delta = \frac{\delta^{oo}}{\lambda^*})$ is a feasible solution to

$$C(y, w|NI) = \min \Big\{ wx \ : X\mu \leq x,\ Y\mu \geq y,\ \mu = \delta\gamma,\ e\gamma = 1,\ \gamma \in \{0,1\}^n,\ \delta \leq 1 \Big\} \tag{A7}$$

with the objective function value $\frac{w x^{oo}}{\lambda^*}$ which is equal to $\frac{C(\lambda^* y, w|ND)}{\lambda^*}$. This implies: $C(y, w|NI) \leq \frac{C(\lambda^* y, w|ND)}{\lambda^*}$. On the other hand, as G-SCES prevails at y, we have $C(y, w|NI) = C(y, w|ND)$ leading in $C(y, w|ND) \leq \frac{C(\lambda^* y, w|ND)}{\lambda^*}$. So far, we have proven:

$$C(\lambda^* y, w|ND) = \lambda^* C(y, w|ND). \tag{A8}$$

Similarly, we get

$$C(\lambda^* y, w|NI) = \lambda^* C(y, w|NI). \tag{A9}$$

Since G-SCES prevails at y, by Theorem 1, we have $C(y, w|NI) = C(y, w|ND)$. So, by (A8) and (A9), we have

$$C(\lambda^* y, w|NI) = C(\lambda^* y, w|ND).$$

Due to (A9) and the proof of Theorem 1,

$$\min_{\lambda \in [1,\infty)} \frac{C(\lambda\lambda^* y, w|V)}{\lambda} = \lambda^* \min_{\lambda \in [1,\infty)} \frac{C(\lambda y, w|V)}{\lambda}. \tag{A10}$$

Hence, $\frac{\lambda^+}{\lambda^*}$ solves $\min_{\lambda \in [1,\infty)} \frac{C(\lambda\lambda^* y, w|V)}{\lambda}$. Furthermore, for each $\tilde{\lambda}$ which solves the last minimization problem, we have $\tilde{\lambda} \geq \frac{\lambda^+}{\lambda^*}$. If this claim is proved, then

$$1 < \frac{\lambda^+}{\lambda^*} = \min \operatorname{argmin} \left\{ \frac{C(\lambda\lambda^* y, w|V)}{\lambda} : \ \lambda \geq 1 \right\}.$$

Hence, $\lambda = 1$ is a feasible but non-optimal solution of the last minimization problem. This implies: $\min_{\lambda \in [1,\infty)} \frac{C(\lambda\lambda^* y, w|V)}{\lambda} < C(\lambda^* y, w|V)$. So,

$$C(\lambda^* y, w|ND) = C(\lambda^* y, w|NI) = \min_{\lambda \in [1,\infty)} \frac{C(\lambda\lambda^* y, w|V)}{\lambda} < C(\lambda^* y, w|V).$$

We have proven (A4). To complete the proof of part (i), we should establish the aforementioned claim: For each $\tilde{\lambda}$ which solves $\min_{\lambda \in [1,\infty)} \frac{C(\lambda\lambda^* y, w|V)}{\lambda}$, we have $\tilde{\lambda} \geq \frac{\lambda^+}{\lambda^*}$. If $\tilde{\lambda}\lambda^* \geq 1$, then due to (A10), $\lambda = \tilde{\lambda}\lambda^*$ solves the model RAC_+, and so $\tilde{\lambda}\lambda^* \geq \lambda^+$. This proves the claim. If $\tilde{\lambda}\lambda^* < 1$, then according to

$$\min_{\lambda \in [1,\infty)} \frac{C(\lambda\lambda^* y, w|V)}{\lambda} = \lambda^* \min_{\lambda \in [1,\infty)} \frac{C(\lambda y, w|V)}{\lambda} = \lambda^* \min_{\lambda \in (0,1]} \frac{C(\lambda y, w|V)}{\lambda},$$

the scalar $\lambda = \tilde{\lambda}\lambda^*$ solves the model RAC_-, and so $\tilde{\lambda}\lambda^* \leq \lambda^-$. This leads in $\lambda^* \leq \tilde{\lambda}\lambda^* \leq \lambda^-$ which is an obvious contradiction.

Part (ii): Assume G-CES prevails at y. Then,

$$C(y, w|NI) = C(y, w|ND) = C(y, w|V).$$

We show that for sufficiently small $\epsilon > 0$ values, the GES status of $(1 \pm \epsilon)y$ is *not* G-CES. Notice that here we implicitly assume $(1 \pm \epsilon)y$ is feasible output in T_V; otherwise there is nothing to prove. First, we work on $(1 + \epsilon)y$. By indirect proof, assume that:

$$\exists\, \epsilon_0 > 0 \ \text{ such that } \quad \begin{aligned} &C((1+\epsilon)y, w|NI) = C((1+\epsilon)y, w|ND) \\ &= C((1+\epsilon)y, w|V), \quad \forall \epsilon \in [0, \epsilon_0]. \end{aligned} \tag{A11}$$

Consider $\epsilon \in (0, \epsilon_0]$ arbitrary. Furthermore, consider two problems (A7) and

$$C((1+\epsilon)y, w|NI) = \min\Big\{wx \quad : X\mu \leq x,\, Y\mu \geq (1+\epsilon)y,\, x \geq 0, \quad \mu = \delta\gamma,\;\, e\gamma = 1,\, \gamma \in \{0,1\}^n,\, \delta \leq 1\}. \tag{A12}$$

If $(x^*, \mu^*, \gamma^*, \delta^*)$ is an optimal solution of (A12), then $\frac{\delta^*}{1+\epsilon} \leq 1$, and so, $(x = \frac{x^*}{1+\epsilon}, \mu = \frac{\mu^*}{1+\epsilon}, \gamma = \gamma^*, \delta = \frac{\delta^*}{1+\epsilon})$ is a feasible solution to (A7), with the objective function value $\frac{wx^*}{1+\epsilon}$ which is equal to $\frac{C((1+\epsilon)y,w|NI)}{1+\epsilon}$. This leads in $C(y, w|NI) \leq \frac{C((1+\epsilon)y,w|NI)}{1+\epsilon}$.

Conversely, assume that $(x^{**}, \mu^{**}, \gamma^{**}, \delta^{**})$ is an optimal solution of (A7). If $(1+\epsilon)\delta^{**} \leq 1$, then $(x = (1+\epsilon)x^{**}, \mu = (1+\epsilon)\mu^{**}, \gamma = \gamma^{**}, \delta = (1+\epsilon)\delta^{**})$ is a feasible solution to (A12), with the objective function value $(1+\epsilon)wx^{**}$ which is equal to $(1+\epsilon)C(y, w|NI)$. This leads in $C\Big((1+\epsilon)y, w|NI\Big) \leq (1+\epsilon)C(y, w|NI)$. If $(1+\epsilon)\delta^{**} \geq 1$, then $(x = (1+\epsilon)x^{**}, \mu = (1+\epsilon)\mu^{**}, \gamma = \gamma^{**}, \delta = (1+\epsilon)\delta^{**})$ is a feasible solution to:

$$C((1+\epsilon)y, w|ND) = \min\Big\{wx \quad : X\mu \leq x,\, Y\mu \geq (1+\epsilon)y,\, x \geq 0, \quad \mu = \delta\gamma,\;\, e\gamma = 1,\, \gamma \in \{0,1\}^n,\, \delta \geq 1\},$$

with the objective function value $(1+\epsilon)wx^{**}$ which is equal to $(1+\epsilon)C(y, w|NI)$. This leads in $C\Big((1+\epsilon)y, w|NI\Big) = C\Big((1+\epsilon)y, w|ND\Big) \leq (1+\epsilon)C(y, w|NI)$; the equality comes from (A11).

Hence,

$$C\Big((1+\epsilon)y, w|NI\Big) = (1+\epsilon)C(y, w|NI). \tag{A13}$$

Now let $\{\epsilon_\nu\}_{\nu=1}^{\infty}$ be a decreasing sequence in $(0, \epsilon_0)$ with $\epsilon_\nu \to 0$ as $\nu \to \infty$. By (A11) and (A13), for each $\nu \in \mathbb{N}$ there exists some $k_\nu \in J$ such that $y_{k_\nu} \geq (1+\epsilon_\nu)y$ and

$$wx_{k_\nu} = C((1+\epsilon_\nu)y, w|NI) = (1+\epsilon_\nu)C(y, w|NI).$$

By choosing an appropriate subsequence, without loss of generality, we assume that k_ν is independent of ν. That is, there exists some $k \in J$ such that

$$y_k \geq (1+\epsilon_\nu)y, \quad wx_k = (1+\epsilon_\nu)C(y, w|NI), \quad \forall \nu \in \mathbb{N}.$$

Since $y_k \geq (1+\epsilon_\nu)y > y$, by setting $\lambda_k := \max\limits_r \dfrac{y_r}{y_{rk}}$, we have $\lambda_k < 1$ and $\lambda_k y_k \geq y$. Now, by setting $x = \lambda_k x_k$, we derive a feasible solution to the model corresponding to $C(y, w|NI)$ (model (A7)) with the objective function value $wx = \lambda_k wx_k = \lambda_k(1+\epsilon_\nu)C(y, w|NI)$. For sufficiently small $\epsilon_\nu > 0$, we have $\lambda_k(1+\epsilon_\nu) < 1$, and so $wx < C(y, w|NI)$. We got a feasible solution to the model corresponding to $C(y, w|NI)$ at which the objective value is strictly less than $C(y, w|NI)$. This contradiction shows that (A11) does not hold. Therefore, for sufficiently small $\epsilon > 0$ values, the GES status of $(1+\epsilon)y$ is *not* G-CES. It can be proved for $(1-\epsilon)y$ analogously. □

Proof of Theorem 3:

Part (i): We have

$$C(\lambda y, w|V) = \begin{array}{ll} \min & wx \\ s.t. & X\mu = x, \quad Y\mu \geq \lambda y, \\ & \sum_{j\in J} \mu_j = 1, \quad \mu_j \in \{0,1\};\ \forall j \in J. \end{array} \tag{A14}$$

We apply an enumeration approach to obtain a closed formula for $C(\lambda y, w|V)$. According to the third and fourth constraints of (A14), assume $\mu_k = 1$, for some k, and $\mu_j = 0$ for each $j \neq k$. Then, $x_k = x$ and $y_k \geq \lambda y$, and wx_k should be minimized. From the last inequality, we get $0 < \lambda \leq \min_r \frac{y_{rk}}{y_r}$. So, for $\lambda > 0$,

$$C(\lambda y, w|V) = \min \left\{ wx_k : \ k \in J,\ \lambda \leq \min_r \frac{y_{rk}}{y_r} \right\}.$$

Hereafter, by defining $\theta_k := \min_r \frac{y_{rk}}{y_r}$ for $k = 1, 2, \ldots, n$, without loss of generality, assume:

$$0 < \theta_1 \leq \theta_2 \leq \ldots \leq \theta_n. \tag{A15}$$

Then, we have:

$$C(\lambda y, w|V) = \begin{cases} \min\limits_{k=1,2,\ldots,n} wx_k & \text{if } \lambda \in (0, \theta_1] \\ \min\limits_{k=2,3,\ldots,n} wx_k & \text{if } \lambda \in (\theta_1, \theta_2] \\ \quad\vdots & \quad\vdots \\ wx_n & \text{if } \lambda \in (\theta_{n-1}, \theta_n] \end{cases} \tag{A16}$$

Part (ii): Now, we focus on $C(\lambda y, w|NI)$, whereby:

$$C(\lambda y, w|NI) = \begin{array}{ll} \min & wx \\ s.t. & X\mu = x, \quad Y\mu \geq \lambda y, \\ & \mu_j = \delta\gamma_j, \quad \gamma_j \in \{0,1\};\ \forall j \in J, \quad \sum_{j\in J} \gamma_j = 1,\ 0 \leq \delta \leq 1. \end{array} \tag{A17}$$

We apply an enumeration approach to obtain a closed formula for $C(\lambda y, w|NI)$. According to the third and fourth constraints of (A17), assume $\mu_k > 0$ and $\mu_j = 0$ for each $j \neq k$. Then, we arrive at the following problem:

$$\begin{array}{ll} \min & \mu_k wx_k \\ s.t. & \mu_k y_k \geq \lambda y, \quad \mu_k \leq 1. \end{array} \tag{A18}$$

Here, μ_k is the only variable of (A18). Notice that $\lambda > 0$ is a parameter. It is not difficult to see that (A18) is feasible if and only if $0 < \lambda \leq \min_r \frac{y_{rk}}{y_r}$. Furthermore, from the constraints of (A18), $\mu_k \geq \lambda \frac{y_r}{y_{rk}}$ for each $r \in \{1, 2, \ldots, s\}$. So, when (A18) is feasible, the smallest value of μ_k is equal to

$\mu_k^* = \lambda \max\limits_r \dfrac{y_r}{y_{rk}}$. Therefore, for $\lambda > 0$, we get:

$$C(\lambda y, w|NI) = \min\left\{\lambda(wx_k)\max_r \frac{y_r}{y_{rk}} :\ k \in J,\ \lambda \leq \min_r \frac{y_{rk}}{y_r}\right\}.$$

Notice that $\lambda \leq \min\limits_r \dfrac{y_{rk}}{y_r}$ if and only if $\mu_k^* = \lambda \max\limits_r \dfrac{y_r}{y_{rk}} \leq 1$. Now, taking (A15) into account, we obtain:

$$C(\lambda y, w|NI) = \begin{cases} \min\limits_{k=1,2,\ldots,n} \dfrac{\lambda w x_k}{\theta_k} & \text{if } \lambda \in (0, \theta_1] \\ \min\limits_{k=2,3,\ldots,n} \dfrac{\lambda w x_k}{\theta_k} & \text{if } \lambda \in (\theta_1, \theta_2] \\ \vdots & \vdots \\ \frac{\lambda w x_n}{\theta_n} & \text{if } \lambda \in (\theta_{n-1}, \theta_n] \end{cases} \tag{A19}$$

Part (iii): For getting a closed formula for $C(\lambda y, w|ND)$, which is as follows:

$$\begin{aligned} C(\lambda y, w|ND) = \quad & \min \quad wx \\ & s.t. \quad X\mu = x, \quad Y\mu \geq \lambda y, \\ & \qquad\ \ \mu_j = \delta\gamma_j, \quad \gamma_j \in \{0,1\};\ \forall j \in J, \quad \sum_{j \in J} \gamma_j = 1,\ \delta \geq 1. \end{aligned} \tag{A20}$$

according to the third and fourth constraints of (A20), by assuming $\mu_k > 0$ and $\mu_j = 0$ for each $j \neq k$, we arrive at the following problem:

$$\begin{aligned} & \min \quad \mu_k w x_k \\ & s.t. \quad \mu_k y_k \geq \lambda y, \quad \mu_k \geq 1. \end{aligned} \tag{A21}$$

Here also, μ_k is the only variable of (A21). According to the constraints of (A18), the optimal value of μ_k is as follows:

$$\mu_k^* = \max\left\{1, \lambda \max_r \frac{y_r}{y_{rk}}\right\} = \max\{1, \frac{\lambda}{\theta_k}\}.$$

Therefore, for $\lambda > 0$, we obtain:

$$C(\lambda y, w|ND) = \min\left\{(wx_k)\max\{1, \frac{\lambda}{\theta_k}\} :\ k \in J\right\}. \tag{A22}$$

If $0 < \lambda \leq \theta_1 \leq \theta_2 \ldots \leq \theta_n$, then $\max\{1, \frac{\lambda}{\theta_k}\} = 1$ for each $k \in J$, and hence $C(\lambda y, w|ND) = \min\{wx_k :\ k \in J\}$. If $\theta_1 < \lambda \leq \theta_2 \leq \theta_3 \ldots \leq \theta_n$, then $\max\{1, \frac{\lambda}{\theta_1}\} = \frac{\lambda}{\theta_1}$ and $\max\{1, \frac{\lambda}{\theta_k}\} = 1$ for each $k \in J \setminus \{1\}$, and hence $C(\lambda y, w|ND) = \min\left\{\frac{\lambda(wx_1)}{\theta_1}, wx_2, wx_3, \ldots, wx_n\right\}$. By following the line of reasoning, taking (A15) into account, we have:

$$C(\lambda y, w|ND) = \begin{cases} \min\{wx_1, wx_2, \ldots, wx_n\}, & \text{if } \lambda \in (0, \theta_1] \\ \min\{\lambda \frac{wx_1}{\theta_1}, wx_2, \ldots, wx_n\}, & \text{if } \lambda \in (\theta_1, \theta_2] \\ \min\{\lambda \frac{wx_1}{\theta_1}, \lambda \frac{wx_2}{\theta_2}, wx_3, \ldots, wx_n\}, & \text{if } \lambda \in (\theta_2, \theta_3] \\ \quad \vdots & \quad \vdots \\ \min\{\lambda \frac{wx_1}{\theta_1}, \ldots, \lambda \frac{wx_{n-1}}{\theta_{n-1}}, wx_n\}, & \text{if } \lambda \in (\theta_{n-1}, \theta_n] \end{cases} \tag{A23}$$

Proof of Theorem 4:

Part (i): According to the closed formulas (A16) and (A23) (Summarized in Theorem 3), we have $C(\lambda y, w|V) = C(\lambda y, w|ND)$ for each $\lambda \in (0, \theta_1)$. Furthermore, for each $\lambda \in (0, \theta_1)$, we get $\frac{\lambda}{\theta_k} < 1$ for each $k = 1, 2, \ldots, n$, and hence by (A19) and (A23), $C(\lambda y, w|NI) < C(\lambda y, w|ND)$. Now, the proof of part (i) is complete thanks to Theorem 1.

Part (ii): For each $\lambda \in (\theta_1, \theta_2)$, we get $\frac{\lambda}{\theta_k} < 1$ for each $k = 2, 3, \ldots, n$. On the other hand, from the assumption, $\pi_2 \leq \frac{\lambda}{\theta_1} wx_1$. Hence,

$$C(\lambda y, w|NI) = \min_{k=2,3,\ldots,n} \frac{\lambda wx_k}{\theta_k} < \min_{k=2,3,\ldots,n} (wx_k) = C(\lambda y, w|V) = \pi_2 \leq \frac{\lambda}{\theta_1} wx_1,$$

and

$$C(\lambda y, w|ND) = \min\{\frac{\lambda}{\theta_1} wx_1, \pi_2\} = \pi_2 > C(\lambda y, w|NI).$$

In summary, $C(\lambda y, w|NI) < C(\lambda y, w|ND) = \pi_2 = C(\lambda y, w|V)$. Now, the proof of part (ii) is complete by Theorem 1.

Part (iii): For each $\lambda \in (\theta_2, \theta_3)$, we get $\frac{\lambda}{\theta_k} < 1$ for each $k = 3, 4, \ldots, n$. On the other hand, from the assumption, $\pi_3 \leq \frac{\lambda}{\theta_1} wx_1$ and $\pi_3 \leq \frac{\lambda}{\theta_2} wx_2$. Hence,

$$C(\lambda y, w|NI) = \min_{k=3,4,\ldots,n} \frac{\lambda wx_k}{\theta_k} < \min_{k=3,4,\ldots,n} (wx_k) = C(\lambda y, w|V) = \pi_3,$$

and

$$C(\lambda y, w|ND) = \min\{\frac{\lambda}{\theta_1} wx_1, \frac{\lambda}{\theta_2} wx_2, \pi_3\} = \pi_3 > C(\lambda y, w|NI).$$

In summary, $C(\lambda y, w|NI) < C(\lambda y, w|ND) = \pi_3 = C(\lambda y, w|V)$. Now, the proof of part (iii) is complete due to Theorem 1.

Part (iv): The proof of this part is similar to that of part (iii) and is hence omitted. $\square$

Proof of Theorem 5:

Part (i): From the assumption, $\pi_2 > \frac{\lambda}{\theta_1} wx_1$, leading to $C(\lambda y, w|ND) < C(\lambda y, w|V)$. Furthermore, from $h_2 < \gamma_1$, there exists some $i \geq 2$ such that $\frac{wx_i}{\theta_i} < \frac{wx_1}{\theta_1}$. Hence,

$$C(\lambda y, w|ND) = \min\{\frac{\lambda}{\theta_1}wx_1, \pi_2\} = \frac{\lambda}{\theta_1}wx_1 > \frac{\lambda}{\theta_i}wx_i \geq C(\lambda y, w|NI).$$

So, $C(\lambda y, w|NI) < C(\lambda y, w|ND) < C(\lambda y, w|V)$. Now, the proof of part (ii) is complete by Theorem 1.

Part (ii): From the assumption, $\min\{\frac{\lambda}{\theta_1}wx_1, \frac{\lambda}{\theta_2}wx_2\} < \pi_3$, leading to $C(\lambda y, w|ND) < C(\lambda y, w|V)$. Furthermore, there exists some $i \geq 3$ such that

$$C(\lambda y, w|NI) \leq \frac{\lambda}{\theta_i}wx_i < \min\{\frac{\lambda}{\theta_1}wx_1, \frac{\lambda}{\theta_2}wx_2\}.$$

Hence,

$$C(\lambda y, w|ND) = \min\{\frac{\lambda}{\theta_1}wx_1, \frac{\lambda}{\theta_2}wx_2, \pi_3\} = \min\{\frac{\lambda}{\theta_1}wx_1, \frac{\lambda}{\theta_2}wx_2\} > C(\lambda y, w|NI).$$

Therefore, $C(\lambda y, w|NI) < C(\lambda y, w|ND) < C(\lambda y, w|V)$, and the proof of part (ii) is complete by Theorem 1.

Part (iii): The proof of this part is similar to that of part (ii) and is hence omitted. □

Proof of Theorem 7:

Part (i): Since $\lambda v_{n-1} < wx_n$, due to the closed formulas expressed in Theorem 3, we have $C(\lambda y, w \mid ND) = \lambda v_{n-1}$. On the other hand, $\lambda \nu_{n-1} < \lambda \frac{wx_n}{\theta_n} = C(\lambda y, w \mid NI)$. Furthermore, $C(\lambda y, w \mid NI) = \frac{\lambda}{\theta_n}wx_n < wx_n = C(\lambda y, w \mid V)$. In summary, $C(\lambda y, w \mid ND) < C(\lambda y, w \mid NI) < C(\lambda y, w \mid V)$, and (i) is proved by applying Theorem 2.

Part (ii): From $\lambda v_{n-2} < \min\{wx_{n-1}, wx_n\}$, we get $C(\lambda y, w \mid ND) = \lambda v_{n-2}$. On the other hand, $\lambda \nu_{n-2} < \min\{\frac{\lambda}{\theta_{n-1}}wx_{n-1}, \frac{\lambda}{\theta_n}wx_n\} = C(\lambda y, w \mid NI)$. Furthermore, $C(\lambda y, w \mid NI) = \min\{\frac{\lambda}{\theta_{n-1}}wx_{n-1}, \frac{\lambda}{\theta_n}wx_n\} < \min\{wx_{n-1}, wx_n\} = C(\lambda y, w \mid V)$. In summary, $C(\lambda y, w \mid ND) < C(\lambda y, w \mid NI) < C(\lambda y, w \mid V)$, and (ii) is proved according to Theorem 2.

Part (iii): This part can be proved similar to part (ii). □

B Appendix B. Second experiment: The electricity distribution sector

The data, extracted from Haghighatpisheh et al. (2022) (Table 1, p. 1467), contains 39 DMUs with two inputs and three outputs. The cost vector is fixed across all units as $w = (1.5, 0.8, 1)$. Table B1 reports the cost measures and classifications. Table B2 presents the quantities used in the stability analysis. When performing the stability analysis, note that after obtaining the θ_k values, you should sort the DMUs based on their θ values and then work with the new numbering accordingly. In Table B1, the first column represents the DMU numbers; the next three columns show the nonconvex cost measures; the following column gives the GES class of units; the next three columns contain the convex cost function values; and the final column shows the classification of units according to their convex economies of scale. In this table, certain numerical values have been rounded to improve the overall fit within the page layout. This rounding facilitates clearer presentation without compromising the integrity of the data. A detailed description of Table B2 is provided in the subsequent paragraphs.

Figures B1 and B2 represent the statistics and distributions of the DMUs across the various GES and convex ES classes. This experiment also highlights that if the problem is *nonconvex in essence*, convex technology does not reveal the ES properly. Here, more than *56%* of the units fall into the new classes introduced in the present paper, G-SIES and G-SDES.

At the beginning of Section 5, we provided a detailed economic interpretation of the proposed global ES classes, highlighting that they identify situations in which a richer set of managerial choices is available regarding the scale of activity of the firm. In particular, the proposed classification captures cases where both optimal and suboptimal scale adjustments may exist in the expansion and contraction directions. For example, in the current empirical illustration, both DMUs 1 and 2 are located in the increasing ES region (in the classical sense); however, they exhibit substantially different scale adjustment opportunities that cannot be distinguished by the local ES classification. DMU 2 has a G-SIES status, meaning that the scales associated with lower average costs are located on both the contraction and expansion sides of its current scale. In contrast, for DMU 1, which has a G-IES status, all scales leading to lower average costs are located on the expansion side, while contraction of the production scale does not result in any improvement in average cost. In some cases, although increasing the scale beyond the current output level can reduce average cost, contraction of the production scale may represent a more practical or feasible alternative for the firm. For instance, in electricity distribution systems, although expanding the network scale may reduce average costs through economies of scale, contraction may be a more practical alternative when electricity demand declines, existing network capacity becomes excessive, or investment constraints limit further expansion. In such cases, reducing the operational scale (e.g., through network consolidation, capacity optimization, or decommissioning underutilized facilities) may lead to lower average costs. The proposed global ES classes provide managers with alternative scale adjustment options for achieving lower average costs, including contraction strategies when expansion is not desirable or feasible. In the empirical illustration, this additional diversity in decision-making is observed for 22 out of the 39 examined DMUs, whereas the remaining DMUs do not exhibit such alternative scale adjustment opportunities.

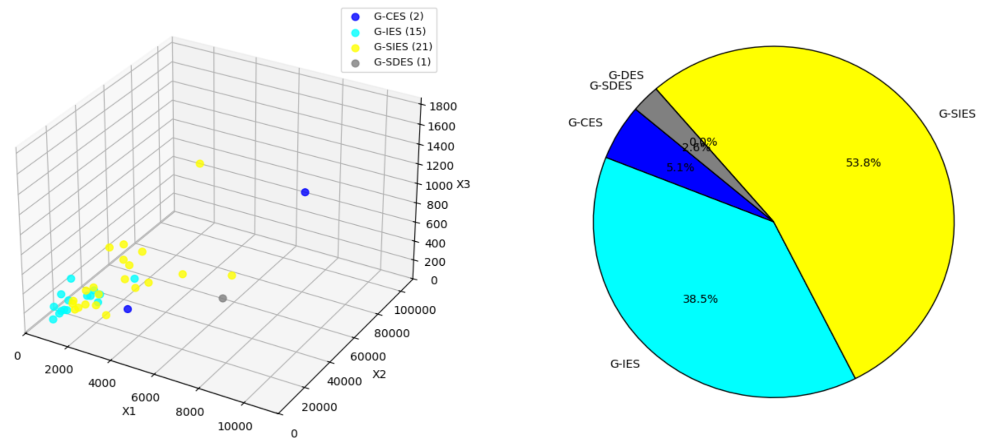

Figure B1: Distribution of DMUs by (nonconvex) GES classification in the electricity distribution data set

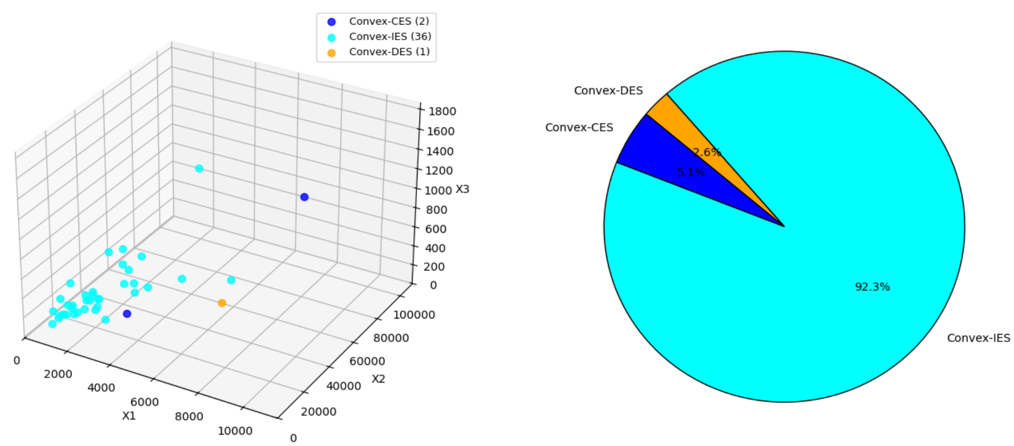

Figure B2: Distribution of DMUs by Convex-ES classification in the electricity distribution data set

We perform two stability analyses by invoking one of the DMUs from the electricity distribution data set. Considering y_1, we provide several λ-intervals corresponding to GES classes at λy. These intervals are derived using the closed-form formulas presented in Subsection 6.3 and summarized in Theorem 3. Based on Theorem 4, we have computed the values of θ_k, π_k, γ_k, h_k, and $\max\{\theta_{t-1}, \frac{\pi_t}{\gamma_1}, \frac{\pi_t}{\gamma_2}, \ldots, \frac{\pi_t}{\gamma_{t-1}}\}$, which are reported in Table B2. In this table, the first column lists the original DMU numbers, followed by the sorted unit indices in the second column. The subsequent four columns report the values of θ_k, π_k, γ_k, and h_k, respectively. The seventh column denotes the maximum value $\max_k := \max\{\theta_{k-1}, \frac{\pi_k}{\gamma_1}, \frac{\pi_k}{\gamma_2}, \ldots, \frac{\pi_k}{\gamma_{k-1}}\}$ as described and applied in Theorem 4. According to this table and the theorem, the G-IES classification prevails at λy_1 for any

λ in the open intervals whose lower and upper bounds are listed in the last two columns of Table B2. Figure B3 provides a schematic illustration of these intervals. For instance, G-IES prevails at $\lambda y_1 = (\lambda \times 819, \lambda \times 6818, \lambda \times 587)$ for any $\lambda \in (4.337958, 4.887743)$. Note that these intervals are well-defined only when $\max\{\theta_{t-1}, \frac{\pi_t}{\gamma_1}, \frac{\pi_t}{\gamma_2}, \ldots, \frac{\pi_t}{\gamma_{t-1}}\} < \theta_t$. This condition explains the absence of intervals corresponding to some values of k in Figure B3 and Table B2.

Additional stability intervals for G-IES at DMU_1 can be derived from Table B2. Furthermore, analogous analyses can be conducted for the G-SIES and G-SDES classes by applying Theorems 5 and 7, respectively. While the present study focuses on DMU_1, similar procedures apply to any other output vectors in the data set.

DMU	C_{NI}	C_{ND}	C_V	NC-Class	C_V^c	C_{ND}^c	C_{NI}^c	C-Class
1	9880.96	11269.6	11269.6	G-IES	11269.6	11269.6	9880.96	IES^c
2	11533.45	13154.32	16314.7	G-SIES	12862.35	12862.35	11533.45	IES^c
3	12472.87	14940.49	16314.7	G-SIES	13913.60	13913.60	12472.87	IES^c
4	5432.83	14501.7	14501.7	G-IES	9148.831	9148.830	5432.83	IES^c
5	22518.48	27563.57	43256	G-SIES	21272.59	21272.59	20900.10	IES^c
6	12851.10	16225.66	16314.7	G-SIES	14375.76	14375.76	12661.99	IES^c
7	3887.29	12433.9	12433.9	G-IES	8621.68	8621.68	3779.68	IES^c
8	11271.01	14056.71	16314.7	G-SIES	11724.92	11724.92	10046.01	IES^c
9	8290.21	12506.96	12506.96	G-IES	10182.48	10182.48	8027.49	IES^c
10	6889.28	10582.81	11269.6	G-SIES	9575.29	9575.29	6708.93	IES^c
11	14800.42	16314.7	16314.7	G-IES	15720.09	15720.09	14238.17	IES^c
12	48295.6	48295.6	48295.6	G-CES	48295.6	48295.6	48295.6	CES^c
13	29793.90	33163.10	43256	G-SIES	27525.11	27525.11	27273.42	IES^c
14	7506.16	9168	9168	G-IES	9168	9168	6835.96	IES^c
15	15596.75	18666.8	18666.8	G-IES	16999.10	16999.10	15596.75	IES^c
16	17924.84	21739.31	43256	G-SIES	17674.64	17674.64	16681.73	IES^c
17	3723.75	12986.52	12986.52	G-IES	8666.41	8666.41	3723.75	IES^c
18	3520.02	11127.3	11127.3	G-IES	8608.90	8608.90	3520.02	IES^c
19	15940.2	15940.2	15940.2	G-CES	15940.2	15940.2	15940.2	CES^c
20	38131.20	30006.39	40316.6	G-SDES	36207.48	30006.39	36207.48	DES^c
21	3727.18	9769.7	9769.7	G-IES	8366.44	8366.45	3298.61	IES^c
22	7296.61	9307.87	11269.6	G-SIES	9232.76	9232.76	6625.23	IES^c
23	10406.65	13139.33	14461.8	G-SIES	11344.63	11344.63	9634.80	IES^c
24	5966.58	13061.9	13061.9	G-IES	8953.91	8953.91	5593.67	IES^c
25	12161.01	14853.43	16314.7	G-SIES	12181.21	12181.21	10695.48	IES^c
26	7673.87	12789.92	12789.92	G-IES	10210.65	10210.65	7673.87	IES^c
27	6372.26	10620.54	11269.6	G-SIES	9586.15	9586.15	6372.26	IES^c
28	3107.95	8235.1	8235.1	G-IES	8235.1	8235.1	2803.10	IES^c
29	13672.61	17149.52	17149.52	G-IES	14022.72	14022.72	12613.50	IES^c
30	16427.38	19915.38	39388.3	G-SIES	15771.53	15771.53	14915.58	IES^c
31	12352.07	15776.55	15940.2	G-SIES	12059.14	12059.14	10451.41	IES^c
32	9629.92	11972.01	16314.7	G-SIES	10602.48	10602.48	8601.83	IES^c
33	12949.78	15816.81	15940.2	G-SIES	12165.39	12165.40	10790.41	IES^c
34	14204.59	16200.86	18666.8	G-SIES	15436.93	15436.93	14204.59	IES^c
35	13988.11	16497.69	18666.8	G-SIES	15350.11	15350.11	13746.50	IES^c
36	5681.84	9469.83	11269.6	G-SIES	9254.89	9254.89	5681.83	IES^c
37	7670.98	11997.63	11997.63	G-IES	9982.57	9982.57	7513.08	IES^c
38	17288.64	18639.24	28176.6	G-SIES	15694.91	15694.91	15424.67	IES^c
39	10642.48	13437.08	16314.7	G-SIES	11602.02	11602.02	9924.46	IES^c

Table B1: Classification in the electricity distribution data set

DMU's No.	New No.	θ_k	π_k	γ_k	h_k	$\max_k$	LB	UB
38	1	0.214622642	8145.5	129618.1978	9595.839231	0	0	0.214622642
28	2	0.222222222	8145.5	36654.75	9595.839231	0.214622642	0.214622642	0.222222222
21	3	0.24742268	8968	38923.27083	9595.839231	0.244661333	0.244661333	0.24742268
18	4	0.314204545	8968	34915.15371	9595.839231	0.24742268	0.24742268	0.314204545
7	5	0.360824742	8968	34109.35714	9595.839231	0.314204545	0.314204545	0.360824742
17	6	0.376861397	8968	48842.89058	9595.839231	0.360824742	0.360824742	0.376861397
24	7	0.388316151	8968	33260.26991	9595.839231	0.376861397	0.376861397	0.388316151
4	8	0.398579545	8968	35783.32145	9595.839231	0.388316151	0.388316151	0.398579545
31	9	0.44558992	8968	38890.01799	9595.839231	0.398579545	0.398579545	0.44558992
22	10	0.450171821	8968	34072.5458	9595.839231	0.44558992	0.44558992	0.450171821
36	11	0.514204545	8968	24444.55249	9595.839231	0.450171821	0.450171821	0.514204545
27	12	0.536647727	8968	31753.60508	9595.839231	0.514204545	0.514204545	0.536647727
14	13	0.556701031	8968	16109.18519	9595.839231	0.536647727	0.536647727	0.556701031
33	14	0.575028637	10812	60243.95618	9595.839231	0.671169887		
19	15	0.575028637	10812	27137.77888	9595.839231	0.671169887		
23	16	0.607101947	10812	23510.05472	9595.839231	0.671169887		
32	17	0.625429553	10812	34599.42033	9595.839231	0.671169887		
10	18	0.642611684	10812	28834.67647	9595.839231	0.671169887		
39	19	0.662084765	10812	29774.88841	9595.839231	0.671169887		
8	20	0.703321879	10812	31998.86238	9595.839231	0.671169887	0.671169887	0.703321879
26	21	0.710511364	10812	28980.53579	9595.839231	0.703321879	0.703321879	0.710511364
37	22	0.728522337	10812	23468.73821	9595.839231	0.710511364	0.710511364	0.728522337
9	23	0.759450172	10812	27202.57466	9595.839231	0.728522337	0.728522337	0.759450172
25	24	0.786941581	10812	47956.54585	9595.839231	0.759450172	0.759450172	0.786941581
2	25	0.81875	10812	129715.4198	9595.839231	0.786941581	0.786941581	0.81875
30	26	0.964490263	10812	40543.17755	9595.839231	0.81875	0.81875	0.964490263
1	27	1	10812	10812	9595.839231	0.964490263	0.964490263	1
20	28	1.04696449	16035.5	38120.68162	9595.839231	1.483120607		
29	29	1.063001145	16035.5	26666.95151	9595.839231	1.483120607		
6	30	1.191294387	16035.5	14061.17596	9595.839231	1.483120607		
3	31	1.218181818	16035.5	24153.20896	9595.839231	1.483120607		
34	32	1.251988636	16035.5	25168.75879	9595.839231	1.483120607		
16	33	1.284077892	16035.5	33937.97235	9595.839231	1.483120607		
11	34	1.327605956	16035.5	12078.50863	9595.839231	1.483120607		
35	35	1.342497136	18350	22032.44923	9595.839231	1.697188309		
5	36	1.465063001	18350	29733.53362	9595.839231	1.697188309		
15	37	1.578465063	18350	11625.21771	9595.839231	1.697188309		
13	38	2.252004582	42680	18952.00407	9595.839231	3.947465779		
12	39	4.887743414	46902	9595.839231	9595.839231	4.337957825	4.337957825	4.887743414

Table B2: Stability quantities in the electricity distribution data set

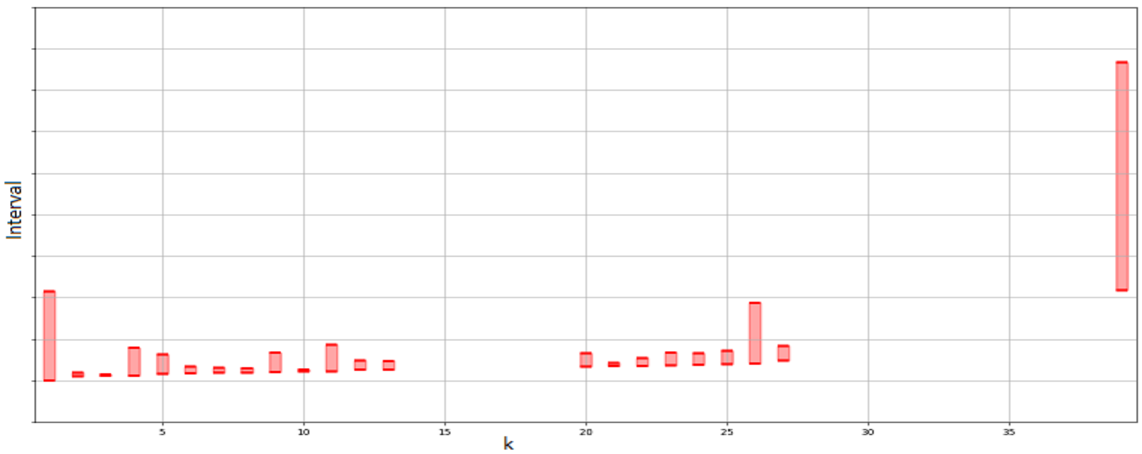

Figure B3: Stability regions in electricity distribution experiment corresponding to G-IES at y_1.

C Appendix C. Mix Adjustment

In economics and operations management, response functions (including the RAC function) are typically defined in a radial (proportional) sense rather than in terms of output-mix adjustments. Consequently, the stability regions of the ES classes derived from such a radial response function are naturally defined with respect to proportional output changes. A natural question arises regarding mixed-output adjustments. One can approach this issue from two directions:

(i) Defining the RAC function in a non-proportional output-mix adjustment sense and then, naturally, deriving stability intervals in the same non-proportional output-mix framework.

(ii) Preserving the radial definition of the RAC function while deriving stability intervals with respect to non-proportional output-mix adjustments.

Direction (i) is completely different from our current study and can be regarded as a separate research problem. Of course, the first issue that should be addressed is whether a mix-based RAC function is meaningful and useful in practice, among other considerations.

However, direction (ii) can be pursued in a manner analogous to our results. Indeed, if one changes the output vector y to the nonnegative vector $y+\beta$ (here, β is a parameter), then, similarly to the proof of Theorem 3, it can be shown that

$$
\begin{aligned}
C(y+\beta, w|V) &= \min\left\{wx_k : \ k \in J,\ \beta \le y_k - y\right\},\\
C(y+\beta, w|NI) &= \min\left\{(wx_k)\max_r \frac{y_r+\beta_r}{y_{rk}} : \ k \in J,\ \beta \le y_k - y\right\},\\
C(y+\beta, w|ND) &= \min\left\{(wx_k)\max\{1, \max_r \frac{y_r+\beta_r}{y_{rk}}\} : \ k \in J\right\}.
\end{aligned}
$$

Now, the stability analysis can be done via these closed-form formulas by following a procedure similar to that used in Theorems 4 to 7.